HMMT Spring 2021

March 06, 2021

Team Round

1. **[40]** Let a and b be positive integers with $a > b$. Suppose that

$$\sqrt{\sqrt{a}+\sqrt{b}}+\sqrt{\sqrt{a}-\sqrt{b}}$$

is a integer.

(a) Must $\sqrt{a}$ be an integer?

(b) Must $\sqrt{b}$ be an integer?

2. **[50]** Let ABC be a right triangle with $\angle A = 90^\circ$. A circle ω centered on BC is tangent to AB at D and AC at E. Let F and G be the intersections of ω and BC so that F lies between B and G. If lines DG and EF intersect at X, show that $AX = AD$.

3. **[50]** Let m be a positive integer. Show that there exists a positive integer n such that each of the $2m+1$ integers

$$2^n - m, 2^n - (m-1), \ldots, 2^n + (m-1), 2^n + m$$

is positive and composite.

4. **[60]** Let k and n be positive integers and let

$$S = \{(a_1, \ldots, a_k) \in \mathbb{Z}^k \mid 0 \le a_k \le \cdots \le a_1 \le n, a_1 + \cdots + a_k = k\}.$$

Determine, with proof, the value of

$$\sum_{(a_1,\ldots,a_k)\in S} \binom{n}{a_1}\binom{a_1}{a_2}\cdots\binom{a_{k-1}}{a_k}$$

in terms of k and n, where the sum is over all k-tuples $(a_1, \ldots, a_k)$ in S.

5. **[60]** A convex polyhedron has n faces that are all congruent triangles with angles 36°, 72°, and 72°. Determine, with proof, the maximum possible value of n.

6. **[70]** Let $f(x) = x^2 + x + 1$. Determine, with proof, all positive integers n such that $f(k)$ divides $f(n)$ whenever k is a positive divisor of n.

7. **[70]** In triangle ABC, let M be the midpoint of BC and D be a point on segment AM. Distinct points Y and Z are chosen on rays $\overrightarrow{CA}$ and $\overrightarrow{BA}$, respectively, such that $\angle DYC = \angle DCB$ and $\angle DBC = \angle DZB$. Prove that the circumcircle of $\triangle DYZ$ is tangent to the circumcircle of $\triangle DBC$.

8. **[80]** For each positive real number α, define

$$\lfloor \alpha\mathbb{N} \rfloor := \{\lfloor \alpha m \rfloor \mid m \in \mathbb{N}\}.$$

Let n be a positive integer. A set $S \subseteq \{1, 2, \ldots, n\}$ has the property that: for each real $\beta > 0$,

$$\text{if } S \subseteq \lfloor \beta\mathbb{N} \rfloor, \text{ then } \{1, 2, \ldots, n\} \subseteq \lfloor \beta\mathbb{N} \rfloor.$$

Determine, with proof, the smallest possible size of S.

9. **[90]** Let scalene triangle ABC have circumcenter O and incenter I. Its incircle ω is tangent to sides BC, CA, and AB at D, E, and F, respectively. Let P be the foot of the altitude from D to EF, and let line DP intersect ω again at $Q \neq D$. The line OI intersects the altitude from A to BC at T. Given that $OI \parallel BC$, show that $PQ = PT$.

10. **[100]** Let $n > 1$ be a positive integer. Each unit square in an $n \times n$ grid of squares is colored either black or white, such that the following conditions hold:

- Any two black squares can be connected by a sequence of black squares where every two consecutive squares in the sequence share an edge;
- Any two white squares can be connected by a sequence of white squares where every two consecutive squares in the sequence share an edge;
- Any 2×2 subgrid contains at least one square of each color.

Determine, with proof, the maximum possible difference between the number of black squares and white squares in this grid (in terms of n).