HMMT Spring 2021

March 06, 2021

Team Round

1. **[40]** Let a and b be positive integers with $a > b$. Suppose that

$$\sqrt{\sqrt{a}+\sqrt{b}}+\sqrt{\sqrt{a}-\sqrt{b}}$$

is a integer.

(a) Must $\sqrt{a}$ be an integer?

(b) Must $\sqrt{b}$ be an integer?

Proposed by: Daniel Zhu

Answer: $\boxed{\text{(a) Yes (b) No}}$

Solution 1: Let $r = \sqrt{\sqrt{a}+\sqrt{b}}$ and $s = \sqrt{\sqrt{a}-\sqrt{b}}$. We know $r^2 + s^2 = 2\sqrt{a}$ and $r^2 - s^2 = 2\sqrt{b}$. If $r + s$ is an integer k, then

$$\sqrt{a} = \frac{r^2+s^2}{2} = \frac{(r+s)^2+(r-s)^2}{4} = \frac{k^2+4b/k^2}{4},$$

which is rational. Recall that since a is a positive integer, $\sqrt{a}$ is either an integer or an irrational number. (Proof: if $\sqrt{a} = p/q$ for relatively prime positive integers p, q, then p^2/q^2 is an integer, which implies $q = 1$.) Thus the answer to part (a) is yes.

The answer to part (b) is no because

$$(2 \pm \sqrt{2})^2 = 6 \pm \sqrt{32},$$

meaning that setting $a = 36$ and $b = 32$ is a counterexample.

Solution 2: Second solution for part (a): Squaring $\sqrt{\sqrt{a}+\sqrt{b}}+\sqrt{\sqrt{a}-\sqrt{b}}$, we see that $2\sqrt{a}+2\sqrt{a-b}$ is an integer. Hence $\sqrt{a-b} = m - \sqrt{a}$ for some rational number m. Squaring both sides of this, we see that $a - b = m^2 - 2m\sqrt{a} + a$, so $\sqrt{a} = \frac{m^2+b}{2m}$, a rational number. As in the first solution, it follows that $\sqrt{a}$ is an integer.

2. **[50]** Let ABC be a right triangle with $\angle A = 90^\circ$. A circle ω centered on BC is tangent to AB at D and AC at E. Let F and G be the intersections of ω and BC so that F lies between B and G. If lines DG and EF intersect at X, show that $AX = AD$.

Proposed by: Krit Boonsiriseth

Solution 1: In all solutions, let O be the center of ω. Then $\angle DOE = 90^\circ$, so $\angle DFE = 45^\circ$, so $\angle DXE = 135^\circ$. Let Γ be the circle centered at A with radius $AD = AE$, and let $X' = \overrightarrow{AX} \cap \Gamma$. Then $\angle DXE = \angle DX'E = 135^\circ$, so $X = X'$.

Solution 2: Let N be the midpoint of arc $FDEG$. Note that $DOEA$ is a square. Also, $DXEN$ is a parallelogram; one way to see this is that by considering inscribed angles, $\angle NEX = \angle NEF = 45^\circ$, $\angle NDX = \angle NDG = 45^\circ$, and $\angle END = 135^\circ$. This means that $\triangle AXD \cong \triangle ONE$, because $AD = OE$, $XD = NE$, and $\angle ADX = \angle OEN$ by considering parallel lines. So $AX = ON = OE = AD$.

Solution 3: Let $Y = DF \cap EG$. Since $GX \perp FY$ and $FX \perp GY$, X is the orthocenter of $\triangle FGY$. Since $(AODE)$ passes through the midpoint of FG, and the feet of altitudes from F and G, $(AODE)$ is the nine-point circle of $\triangle FGY$. Since AO is the diameter of $(AODE)$, it follows that A is the midpoint of XY, so A is the center of $(DXEY)$, so $AX = AD$.

Solution 4: Let $Z = DE \cap FG$, possibly at infinity. Then A and X are on the polar of Z with respect to ω, so $AX \perp BC$. Let $J = AX \cap BC$. Then $(XJFD)$ is cyclic, so $\angle ADX = \angle DFJ = \angle DXA$, so $AX = AD$.

Solution 5: We use complex numbers, with $(FDEG)$ being the unit circle, and $d = -1, e = -i$. As FG is a diameter of the unit circle, we have $g = -f$. We have $a = -1 - i$, from either intersecting the tangents to the unit circle at D and E or noting that $ADOE$ is a square.

Now, intersecting the chords DG and EF, we obtain

$$x = \frac{dg(e+f) - ef(d+g)}{dg - ef} = \frac{f(i+f) + if(1-f)}{f + if} = \frac{f - i - i - if}{1+i} = -1 - i + f \cdot \frac{1-i}{1+i}.$$

So,

$$|x - a| = \left| f \cdot \frac{1-i}{1+i} \right| = 1 \cdot \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

So AX and AD have the same length (1 unit), as desired.

3. **[50]** Let m be a positive integer. Show that there exists a positive integer n such that each of the $2m+1$ integers

$$2^n - m, 2^n - (m-1), \ldots, 2^n + (m-1), 2^n + m$$

is positive and composite.

Proposed by: Michael Ren

Solution: Let P be the set of prime divisors of the $2m+1$ numbers

$$2^{m+1} - m, 2^{m+1} - m + 1, \ldots, 2^{m+1} + m.$$

We claim that

$$n = m + 1 + \prod_{p \in P} (p-1)$$

works. To check this, let k be any integer with $|k| \leq m$. We can take some prime $q | 2^{m+1} + k$, as $2^{m+1} + k \geq 2^{m+1} - m \geq 3$. Let $\prod_{p \in P}(p-1) = (q-1)\alpha_q$. Then, applying Fermat's little theorem, we have

$$2^n \equiv 2^{m+1} \cdot (2^{q-1})^{\alpha_q} \equiv 2^{m+1} \equiv -k \pmod{q}$$

Thus, $2^n + k$ is divisible by q and is bigger than q, so it is positive and composite. This works for each of the required k, so we are done.

4. **[60]** Let k and n be positive integers and let

$$S = \{(a_1, \ldots, a_k) \in \mathbb{Z}^k \mid 0 \leq a_k \leq \cdots \leq a_1 \leq n, a_1 + \cdots + a_k = k\}.$$

Determine, with proof, the value of

$$\sum_{(a_1, \ldots, a_k) \in S} \binom{n}{a_1} \binom{a_1}{a_2} \cdots \binom{a_{k-1}}{a_k}$$

in terms of k and n, where the sum is over all k-tuples $(a_1, \ldots, a_k)$ in S.

Proposed by: Milan Haiman

Answer: $\boxed{\binom{k+n-1}{k} = \binom{k+n-1}{n-1}}$

Solution 1: Let

$$T = \{(b_1, \dots, b_n) \mid 0 \le b_1, \dots, b_n \le k, b_1 + \dots + b_n = k\}.$$

The sum in question counts $|T|$, by letting a_i be the number of b_j that are at least i. By stars and bars, $|T| = \binom{k+n-1}{k}$.

One way to think about T is as follows. Suppose we wish to choose k squares in a grid of squares with k rows and n columns, such that each square not in the bottom row has a square below it. If we divide the grid into columns and let b_j be the number of chosen squares in the jth column then we get that T is in bijection with valid ways to choose our k squares.

On the other hand, if we divide the grid into rows, and let a_i be the number of chosen squares in the ith row (counting up from the bottom), then we obtain the sum in the problem. This is because we have $\binom{n}{a_1}$ choices for the squares in the first row, and $\binom{a_{i-1}}{a_i}$ choices for the squares in the ith row, given the squares in the row below, for each $i = 2, \dots, k$.

Solution 2: Define

$$F_k(x, y) = \sum_{n, a_1, \dots, a_k} \binom{n}{a_1} \cdots \binom{a_{k-1}}{a_k} x^n y^{a_1 + \dots + a_k},$$

where the sum is over all nonegative integers $n, a_1, \dots, a_k$ (the nonzero terms have $n \ge a_1 \ge \dots \ge a_k$). Note that we are looking for the coefficient of $x^n y^k$ in $F_k(x, y)$. By first summing over a_k on the inside and using the Binomial theorem, we obtain

$$F_k(x, y) = \sum_{n, a_1, \dots, a_{k-1}} \binom{n}{a_1} \cdots \binom{a_{k-2}}{a_{k-1}} x^n y^{a_1 + \dots + a_{k-1}} (1 + y)^{a_{k-1}}.$$

Now, we repeat this by summing over a_{k-1}, then a_{k-2}, and so on. We obtain

$$F_k(x, y) = \sum_n x^n (1 + y + \dots + y^k)^n.$$

So the answer is just the coefficient y^k in $(1 + y + \dots + y^k)^n$. By stars and bars, this is $\binom{k+n-1}{k}$.

Although we didn't need it, another way to write $F_k(x, y)$ is

$$F_k(x, y) = \frac{1}{1 - x - xy - \dots - xy^k}.$$

Solution 3: Let

$$S(n, k, k') = \{(a_1, \dots, a_k) \mid 0 \le a_k \le \dots \le a_1 \le n, a_1 + \dots + a_k = k'\},$$

and note that $S(n, k, k)$ is the set S in the problem.

Define

$$f(n, k, k') = \sum_{(a_1, \dots, a_k) \in S(n, k, k')} \binom{n}{a_1} \binom{a_1}{a_2} \cdots \binom{a_{k-1}}{a_k}.$$

Now, consider $(a_1, \dots, a_k) \in S(n, k, k')$. We have

$$i a_i \le a_1 + \dots + a_k = k'.$$

So $a_i \le \frac{k'}{i}$. In particular, if $k > k'$, then we have $a_i = 0$ for each $k' < i \le k$. This means that $f(n, k, k') = f(n, k', k')$ if $k > k'$.

Now, let $f(n, k) = f(n, k, k)$ be the answer. Splitting the sum based on the possible values of a_1 gives

$$f(n, k) = \sum_{a_1} \binom{n}{a_1} f(a_1, k-1, k-a_1).$$

For $k > 0$, we must have $a_1 \geq 1$, which means

$$f(a_1, k-1, k-a_1) = f(a_1, k-a_1, k-a_1) = f(a_1, k-a_1).$$

So, we obtain the recurrence

$$f(n, k) = \sum_{a_1} \binom{n}{a_1} f(a_1, k-a_1).$$

Now we claim $f(n, k) = \binom{k+n-1}{k}$. We proceed by induction, with our base case being $k = 1$. The claim is easy to verify for $k = 1$.

In the inductive step, we obtain

$$f(n, k) = \sum_{a_1} \binom{n}{a_1} \binom{k-1}{k-a_1} = \binom{k+n-1}{k},$$

where we applied Vandermonde's identity in the last equality.

5. **[60]** A convex polyhedron has n faces that are all congruent triangles with angles $36°$, $72°$, and $72°$. Determine, with proof, the maximum possible value of n.

Proposed by: Handong Wang

Answer: 36

Solution: Consider such a polyhedron with V vertices, E edges, and $F = n$ faces. By Euler's formula we have $V + F = E + 2$.

Next, note that the number of pairs of incident faces and edges is both $2E$ and $3F$, so $2E = 3F$.

Now, since our polyhedron is convex, the sum of the degree measures at each vertex is strictly less than $360 = 36 \cdot 10$. As all angle measures of the faces of our polyhedron are divisible by 36, the maximum degree measure at a given vertex is $36 \cdot 9 = 324$. On the other hand, the total degree measure at all vertices is the total degree measure over all faces, which is $180F$. Thus we have $180F \leq 324V$, or $10F \leq 18V$.

Putting our three conditions together, we have

$$10F \leq 18V = 18(E + 2 - F) = 9(2E) + 36 - 18F = 9(3F) + 36 - 18F = 9F + 36.$$

Thus $F \leq 36$.

$F = 36$ is attainable by taking a 9-gon antiprism with a 9-gon pyramid attached on the top and the bottom. Thus the answer is 36.

6. **[70]** Let $f(x) = x^2 + x + 1$. Determine, with proof, all positive integers n such that $f(k)$ divides $f(n)$ whenever k is a positive divisor of n.

Proposed by: Milan Haiman

Answer: n can be 1, a prime that is 1 mod 3, or the square of any prime except 3.

Solution: The answer is n can be 1, a prime that is 1 mod 3, or the square of any prime except 3. It is easy to verify that all of these work.

First note that n must be 1 mod 3 since 1 divides n implies $f(1)$ divides $f(n)$.

Next, suppose for sake of contradiction that $n = ab$, with $a > b > 1$. We are given that $f(a)$ divides $f(n)$, which means $f(a)$ divides $f(n) - f(a)$. We can write this as

$$a^2 + a + 1 \mid n^2 + n - a^2 - a = (n - a)(n + a + 1).$$

Since we are working mod $a^2 + a + 1$, we can replace $a + 1$ with $-a^2$, so we have

$$a^2 + a + 1 \mid (n - a)(n - a^2) = a^2(b - 1)(b - a).$$

However, $a^2 + a + 1$ cannot share any factors with a, and $0 < |(b - 1)(b - a)| < a^2 + a + 1$, which is a contradiction.

7. **[70]** In triangle ABC, let M be the midpoint of BC and D be a point on segment AM. Distinct points Y and Z are chosen on rays $\overrightarrow{CA}$ and $\overrightarrow{BA}$, respectively, such that $\angle DYC = \angle DCB$ and $\angle DBC = \angle DZB$. Prove that the circumcircle of $\triangle DYZ$ is tangent to the circumcircle of $\triangle DBC$.

   Proposed by: Joseph Heerens

   Solution 1: We first note that the circumcircles of DBZ and YDC are tangent to BC from our angle criteria. By power of a point, we obtain that M lies on the radical axis of the two circles and clearly D does as well. Therefore, we find that A lies on the radical axis so $AY \cdot AC = AB \cdot AZ$ implying that $BYZC$ is a cyclic quadrilateral.

   Next, by Reim's Theorem on $(BYZC)$ and (DYZ), we get that (DYZ) intersects AB, AC at B', C' where $BC, B'C'$ are parallel. Then a negative homothety maps B to B' and C to C', so (DBC) gets mapped to $(DB'C')$, and we have tangent circles.

   Solution 2: Let (DYZ) intersect AB and AC at B' and C', respectively. We see that $\measuredangle YC'B' = \measuredangle YZB' = \measuredangle YZB = \measuredangle YCB$. Thus, $BC \parallel B'C'$. This means that there exists a negative homothety taking B to B' and C to C' which will map (DBC) to $(DB'C')$ which is also (DYZ).

8. **[80]** For each positive real number α, define

$$\lfloor \alpha\mathbb{N} \rfloor := \{\lfloor \alpha m \rfloor \mid m \in \mathbb{N}\}.$$

   Let n be a positive integer. A set $S \subseteq \{1, 2, \ldots, n\}$ has the property that: for each real $\beta > 0$,

$$\text{if } S \subseteq \lfloor \beta\mathbb{N} \rfloor, \text{ then } \{1, 2, \ldots, n\} \subseteq \lfloor \beta\mathbb{N} \rfloor.$$

   Determine, with proof, the smallest possible size of S.

   Proposed by: Krit Boonsiriseth

   Answer: $\boxed{\lfloor n/2 \rfloor + 1}$

   Solution: For each $k \in \{\lceil n/2 \rceil, \ldots, n\}$, picking $\beta = 1 + 1/k$ gives

$$\lfloor \beta\mathbb{N} \rfloor \cap [n] = [n] \setminus \{k\}$$

   so S must contain k.

   Now we show that $S = \{\lceil n/2 \rceil, \ldots, n\}$ works; this set S has $\lfloor n/2 \rfloor + 1$ elements.

   Suppose β satisfy $S \subseteq \lfloor \beta\mathbb{N} \rfloor$, and suppose for the sake of contradiction that $[n] \not\subseteq \lfloor \beta\mathbb{N} \rfloor$. Since we may increase β by a small amount ε without affecting $\lfloor \beta\mathbb{N} \rfloor \cap [n]$, we may assume β is irrational. Let α satisfy $1/\alpha + 1/\beta = 1$. By Beatty's Theorem, $\lfloor \alpha\mathbb{N} \rfloor$ and $\lfloor \beta\mathbb{N} \rfloor$ are complement sets in $\mathbb{N}$.

Let m be the maximal element of $[n]$ that is not in $\lfloor \beta\mathbb{N} \rfloor$. Then $m = \lfloor k\alpha \rfloor$ for some integer k. Consider $m' = \lfloor 2k\alpha \rfloor \in \{2m, 2m+1\}$, which must be an element of $\lfloor \alpha\mathbb{N} \rfloor$. Clearly, $m' > m$, and since $m < n/2$, $m' \leqslant n$, so m' is also an element of $[n]$ that is not in $\lfloor \beta\mathbb{N} \rfloor$. This contradicts the maximality of m, and we are done.

9. **[90]** Let scalene triangle ABC have circumcenter O and incenter I. Its incircle ω is tangent to sides BC, CA, and AB at D, E, and F, respectively. Let P be the foot of the altitude from D to EF, and let line DP intersect ω again at $Q \neq D$. The line OI intersects the altitude from A to BC at T. Given that $OI \parallel BC$, show that $PQ = PT$.

Proposed by: Carl Schildkraut, Milan Haiman

Solution: Let H be the orthocenter of ΔDEF. We first claim that O, I, H are collinear. We present two proofs.

Proof 1. Invert about ω. Circle (ABC) inverts to a circle with center on OI, but A, B, C invert to the midpoints of EF, FD, DE, respectively, so the nine-point center of ΔDEF is on OI. As this center is the midpoint of IH, we get that H, I, O are collinear. ■

Proof 2. Let Q_A, Q_B, Q_C be the second intersections of the $D-$, $E-$, and $F-$ altitudes, respectively, in ΔDEF with ω. We claim $\Delta Q_AQ_BQ_C$ is homothetic with ΔABC. Indeed, as Q_B is the reflection of H over DF and Q_C is the reflection of H over DE, $DQ_B = DQ_C$, so the perpendicular bisector of Q_BQ_C is line ID. As $ID \perp BC$, $Q_BQ_C || BC$, whence the homothety follows. This homothety takes the incircle to the circumcircle, so it is centered on line OI. However, it also takes the incenter H of $Q_AQ_BQ_C$ to the incenter I of ABC, so it is centered on line IH. So, O, I, H are collinear. ■

As P is the midpoint of QH, it suffices to show that P is on the circle with diameter QH, or that $\angle QTH = 90°$. As $AT \perp \overline{TH} = \overline{IO}$, it suffices to show that Q is on line AT. We also present two proofs of this.

Proof 1. Let D' be the antipode of D, and let AD' intersect BC at X. As X is the A-extouch point, the midpoint M of DX is also the midpoint of BC. We have

$$\frac{OM}{MX} = \frac{ID}{\frac{DX}{2}} = \frac{DD'}{DX}$$

and $\angle OMX = \angle D'DX = 90°$, so D', O, X are collinear, so D' is on line AO. As $QD' || EF$, AQ and AD' are isogonal in $\angle BAC$, so AQ and AO are isogonal, which means Q is on the A-altitude, finishing the proof. ■

Proof 2. Let Γ denote the circumcircle of ΔABC, and let M be the midpoint of arc BC on Γ not containing A.

Lemma. The intersection T' of MD and the A-altitude to BC is on the line through I parallel to BC.

Proof. Let $D' = MA \cap BC$. As $\angle D'BM = \angle CBM = \angle CAM = \angle MAB$, $\triangle D'BM \sim \triangle BAM$, and

$$MI^2 = MB^2 = MD' \cdot MA.$$

Since $AT' \| ID$, we have

$$\frac{MT'}{MD} = \frac{MA}{MI} = \frac{MI}{MD'},$$

so $IT' || DD'$, finishing the proof. ■

By the above lemma, T is on MD. Consider a homothety centered at T that takes D to M. It takes ω to a circle centered on line IT that is tangent to Γ at M; since O is on line IT this circle must be Γ itself. So, T is the exsimilicenter of Γ and ω. By Proof 2 above, T is the center of the homothety which sends $Q_AQ_BQ_C$ to ABC, so T, $Q = Q_A$, and A are collinear, finishing the proof. ■

10. **[100]** Let $n > 1$ be a positive integer. Each unit square in an $n \times n$ grid of squares is colored either black or white, such that the following conditions hold:

- Any two black squares can be connected by a sequence of black squares where every two consecutive squares in the sequence share an edge;
- Any two white squares can be connected by a sequence of white squares where every two consecutive squares in the sequence share an edge;
- Any 2×2 subgrid contains at least one square of each color.

Determine, with proof, the maximum possible difference between the number of black squares and white squares in this grid (in terms of n).

Proposed by: Yuan Yao

Answer: $\boxed{2n + 1 \text{ if } n \text{ is odd, } 2n - 2 \text{ if } n \text{ is even.}}$

Solution: The first two conditions also imply that there can be no 2×2 checkerboards, so the boundary between black squares and white squares is either a lattice path or cycle (if one color encloses the other). Therefore, the set of squares of each color is the interior of a lattice polygon of genus 0 or 1. (In the latter case, the genus-1 color uses all squares on the outer boundary, and the opposite color must be genus-0.)

The third condition requires that the perimeter of each color passes through all $(n-1)^2$ interior lattice points, or else there will be a monochromatic 2×2 subgrid. Hence, by Pick's Theorem, the area of one color is at least $(n-1)^2/2 - 1 = (n^2 - 2n - 1)/2$, and the difference is at most $n^2 - (n^2 - 2n - 1) = 2n + 1$.

For even n, the number of interior lattice points is odd so there is no cycle that only uses them. (In particular, this means that both colors are genus-0.) It is impossible for the perimeter to only go through one boundary point either, so we need to add at least three more boundary points, which means that we lose $2(3/2) = 3$ from the bound for odd n.

Here is one possible set of constructions. Throughout, we'll label the squares as (x, y), for $1 \leq x, y \leq n$:

- For $n = 2$, we color $(2, 2)$ black and the others white.
- For odd values of n, we create a comb shape using black squares. Specifically, the base of the comb will consist of the squares $(i, 2)$, for $i = 2, 3, \ldots, n-1$. The teeth of the comb will be $(2k, j)$, for $k = 1, 2, \ldots, \frac{n-1}{2}$, and $j = 3, 4, \ldots, n-1$.
- For even values of $n > 2$, we make a modified comb shape. The base of the comb will be $(i, 2)$ for $i = 2, 3, \ldots, n$, and the teeth will be $(2k, j)$ for $k = 1, 2, \ldots, \frac{n}{2} - 1$ and $j = 3, 4, \ldots, n-1$. Furthermore, we add the square $(n, 3)$, and the squares $(n-1, 2k+3)$ for $k = 1, 2, \ldots, \frac{n}{2} - 2$.