HMIC 2021

April 24, 2021 – May 2, 2021

1. **[5]** 2021 people are sitting around a circular table. In one move, you may swap the positions of two people sitting next to each other. Determine the minimum number of moves necessary to make each person end up 1000 positions to the left of their original position.

   Proposed by: Milan Haiman

   Answer: $\boxed{1021000}$

   Solution 1: We claim that the answer is $1000 \cdot 1021 = 1021000$. To see how we can obtain this, label the people around the table $s_1, s_2, ..., s_{2021}$. We then start with s_1 and swap them 1000 positions to the left, then we take s_2 and swap them 1000 positions to the left, and so on until we swap s_{1021} 1000 positions to the left. We see that after swapping s_1 1000 positions to the left, we have $s_{1022}, s_{1023}, ..., s_{2021}$ still in order followed by $s_2, s_3, ..., s_{1021}, s_1$. Further, we can show that after moving s_k to the left 1000 times, s_k is to the left of s_{1022} with the order $s_{1022}, s_{1023}, ..., s_{2021}, s_{k+1}, s_{k+2}, ..., s_{1021}, s_1, s_2, ..., s_k$ for $1 \le k \le 1021$. As we never swap any s_k for $1 \le k \le 1021$ other than moving it 1000 times to the left, when the operation is done, $s_1, s_2, ..., s_{1021}$ are all shifted 1000 places to the left followed by $s_{1022}, s_{1023}, ..., s_{2021}$. Therefore, every person is shifted 1000 to the left and this is done in a total of $1000 \cdot 1021$ moves.

   Now, to prove that this is a sufficient lower bound, we note that if a total of M moves are made, then some student will move one seat left M times and one student will move right M times. In order to move every student 1000 seats to the left, we can either have them make at least 1000 total leftward movements or 1021 total rightward movements. This means that there can be at most $\lfloor \frac{M}{1000} \rfloor$ students who get 1000 seats to the left via a net leftward movement and at most $\lfloor \frac{M}{1021} \rfloor$ students who get 1000 seats to the left via a net rightward movement. Thus, we see that

$$\left\lfloor \frac{M}{1000} \right\rfloor + \left\lfloor \frac{M}{1021} \right\rfloor \ge 2021 \implies M \ge 1021 \cdot 1000,$$

   as desired.

   Solution 2: After getting the same construction as above, we can let the students be labeled s_i for $1 \le i \le 2021$, and say that in order for each student to end up 1000 spaces to the left, they are moved left a_i times and right b_i times. Then the condition is trivially equivalent to $a_i - b_i = 1000 \mod 2021$. Further, as each move will move one person right and one person left, we see that $\sum_{i=1}^{2021} a_i$ and $\sum_{i=1}^{2021} b_i$ count the same event. Thus,

$$\sum_{i=1}^{2021} a_i = \sum_{i=1}^{2021} b_i.$$

   Now, let $a_i - b_i = 2021c_i + 1000$ for integer c_i and all $1 \le i \le 2021$. We see that

$$0 = \sum_{i=1}^{2021} a_i - \sum_{i=1}^{2021} b_i = \sum_{i=1}^{2021} (a_i - b_i) = \sum_{i=1}^{2021} (1000 + 2021c_i) = 1000 \cdot 2021 + 2021 \sum_{i=1}^{2021} c_i.$$

   Therefore, we find that

$$\sum_{i=1}^{2021} c_i = -1000.$$

   We then can use the fact that $a_i + b_i \ge |a_i - b_i| = |2021c_i + 1000|$ as both a_i and b_i are nonnegative. Further, the total number of moves is equal to both $\sum_{i=1}^{2021} a_i$ and $\sum_{i=1}^{2021} b_i$ so we can then find that

$$\sum_{i=1}^{2021} \frac{a_i + b_i}{2} \ge \sum_{i=1}^{2021} \frac{|a_i - b_i|}{2} = \sum_{i=1}^{2021} \frac{|2021c_i + 1000|}{2}.$$

Now, let $\sum_{c_i \geq 0} c_i = S$ and $\sum_{c_i<0} c_i = -S - 1000$. Then, we find that

$$\sum_{i=1}^{2021} \frac{|2021c_i + 1000|}{2} = \sum_{c_i \geq 0} \frac{2021c_i + 1000}{2} + \sum_{c_i<0} \frac{-2021c_i - 1000}{2}$$
$$= \frac{2021(2S + 1000)}{2} + 500\left(-2021 + 2\sum_{c_i \geq 0} 1\right).$$

Combining all of this, we find that

$$\sum_{i=1}^{2021} \frac{a_i + b_i}{2} \geq \sum_{i=1}^{2021} \frac{|a_i - b_i|}{2} \geq \frac{2021(2S + 1000)}{2} + 500(-2021 + 2\sum_{c_i \geq 0} 1) = 2021S + 1000 \cdot \sum_{c_i \geq 0} 1.$$

To finish this, we know that if $\sum_{c_i<0} c_i = -S - 1000$, then there are at most $-S - 1000$ values of $c_i < 0$, as the c_i with smallest absolute value is -1. Thus, we find that

$$2021S + 1000 \cdot \sum_{c_i \geq 0} 1 \geq 2021S + 1000(2021 - (S + 1000)) = 1021S + 1000 \cdot 1021.$$

As $S \geq 0$, we find that this makes the total number of moves, $\sum_{i=1}^{2021} \frac{a_i+b_i}{2}$ at least $1000 \cdot 1021$. Therefore, we have our desired lower bound and the result follows.

2. **[7]** Let n be a positive integer. Alice writes n real numbers $a_1, a_2, \dots, a_n$ in a line (in that order). Every move, she picks one number and replaces it with the average of itself and its neighbors (a_n is not a neighbor of a_1, nor vice versa). A number *changes sign* if it changes from being nonnegative to negative or vice versa. In terms of n, determine the maximum number of times that a_1 can change sign, across all possible values of $a_1, a_2, \dots, a_n$ and all possible sequences of moves Alice may make.

   Proposed by: Zhao Yu Ma

   Answer: $\boxed{n-1}$

   Solution: The maximum number is $n-1$. We first prove the upper bound. For simplicity, color all negative numbers red, and all non-negative numbers blue. Let X be the number of color changes among adjacent elements (i.e. pairs of adjacent elements with different colors). It is clear that the following two statements are true:

   (1) When a_1 changes sign, X decreases by 1. If a_1 changes from negative (red) to non-negative (blue), a_2 must have been non-negative (blue), so the first two colors changed from RB to BB. The same applies when a_1 changes from non-negative to negative.

   (2) X cannot increase after a move. Suppose Alice picks a_i for her move where $1 \leq i \leq n$. If a_i does not change sign, then X clearly remains the same. Else, if a_i changes sign and $i = 1$ or n, then X decreases by 1 (from (1)). Finally, if a_i changes sign and $i \neq 1, n$, we have two cases:

   Case 1: a_{i-1}, a_{i+1} are of the same color. If they are both negative (red), then if a_i changes color, it must be from non-negative to negative (blue to red). Thus, the colors change from RBR to RRR and X decreases by 2. The same holds if both a_{i-1}, a_{i+1} are non-negative.

   Case 2: a_{i-1}, a_{i+1} are of of different colors. No matter what the color of a_i is, there is exactly one color change among the three numbers, so X will remain the same.

   Now, since the initial value of X is at most $n-1$, it can decrease by 1 at most $n-1$ times. Hence, a_1 can change signs at most $n-1$ times.

   Now we prove the lower bound by constructing such a sequence inductively. Specifically, we induct on the following statement:

For every $n \geq 2$, there exists a sequence $a_1, a_2, \cdots, a_n$ such that by picking

$$a_1, a_2, a_1, a_3, a_2, a_1, \cdots, a_{n-1}, a_{n-2}, \cdots, a_2, a_1$$

in that order, a_1 changes sign $n-1$ times.

When $n = 2$, we can let a_1 change sign once by starting with the sequence $(1, -3)$, and picking a_1 to obtain $(-1, -3)$, which satisfies the conditions in our statement.

Suppose we have proven the statement for $n-1$. For n, let $a_1, a_2, \cdots, a_{n-1}$ be as defined in our construction for $n-1$ (we shall fix the value of a_n later). After executing the steps $a_1, a_2, a_1, a_3, a_2, a_1, \cdots, a_{n-2}, a_{n-3}, \cdots, a_2, a_1$, a_1 would have changed sign $n-2$ times.

It now remains to pick $a_{n-1}, a_{n-2}, \cdots, a_2, a_1$ in order so that a_1 changes sign one more time. This is always possible as long as a_n is sufficiently large in magnitude and of the opposite sign as a_1. Since the value of a_n has remained unchanged since the start (as we have not picked a_n at all), it suffices to let a_n be a number satisfying the above conditions at the start.

This completes our induction, and we conclude that the maximum number of times that a_1 can change sign is $n-1$.

3. **[8]** Let A be a set of $n \geq 2$ positive integers, and let $f(x) = \sum_{a \in A} x^a$. Prove that there exists a complex number z with $|z| = 1$ and $|f(z)| = \sqrt{n-2}$.

Proposed by: Milan Haiman

Solution 1: Let A consist of the numbers $a_1 < a_2 < \cdots < a_n$. Let $d = a_n - a_1$. Then, note that if we choose z uniformly at random from the set $X = \{x : x^d = -1\}$, then we have

$$f(z) = (z^{a_1} + z^{a_n}) + \sum_{i=2}^{n-1} z^{a_i} = \sum_{i=2}^{n-1} z^{a_i}$$

Then, note that

$$|f(z)|^2 = f(z)f(\bar{z}) = \left(\sum_{i=2}^{n-1} z^{a_i}\right)\left(\sum_{i=2}^{n-1} \bar{z}^{a_i}\right) = n-2 + \sum_{i \neq j, 2 \leq i,j \leq n-1} z^{a_i - a_j}$$

However, since $0 < |a_i - a_j| < d$, for all $2 \leq i, j \leq n-1$ with $i \neq j$, we know that the average value of $z^{a_i - a_j}$ over all $z \in X$ is just 0. Thus, the average value of $|f(z)|^2$ is $n-2$, and thus, there exists z_1 and z_2 on the unit circle such that $|f(z_1)|^2 \leq n-2 \leq |f(z_2)|^2$, and thus by Intermediate Value Theorem, there exists z on the unit circle such that $|f(z)|^2 = n-2$, so we are done.

Solution 2: Note that we have

$$|f(x)|^2 = f(x)f(\bar{x}) = \left(\sum_{a \in A} x^a\right)\left(\sum_{b \in A} \bar{x}^b\right) = n + \sum_{a \neq b} x^{a-b}$$

Let $t = 1 + \max_{a \neq b, a,b \in A} v_2(a-b)$. Let z be chosen uniformly at random from $X = \{x : x^{2^t} = 1, x^{2^{t-1}} = -1\}$, the set of all primitive 2^tth roots of unity. Note that for all distinct $a, b \in A$, we know that the expected value of z^{a-b} is 1 if $v_2(t) \geq t$, -1 if $v_2(a-b) = t-1$, and 0 if $v_2(a-b) < t-1$. However, we know that there exists distinct $\hat{a}, \hat{b} \in A$ such that $v_2(\hat{a} - \hat{b}) = t-1$, so the expected values of $z^{\hat{a}-\hat{b}}$ and $z^{\hat{b}-\hat{a}}$ are both -1. Also, we know that the expected value of every other term in this sum $\sum_{a \neq b} x^{a-b}$ is less than or equal to 0. Thus, the expected value of $|f(z)|^2$ is at most $n-2$. Additionally, since $|f(1)|^2 = n^2$, and then finish with Intermediate Value Theorem.

4. **[10]** Let $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, and $C_1C_2C_3C_4$ be three regular tetrahedra in 3-dimensional space, no two of which are congruent. Suppose that, for each $i \in \{1, 2, 3, 4\}$, C_i is the midpoint of the line segment A_iB_i. Determine whether the four lines A_1B_1, A_2B_2, A_3B_3, and A_4B_4 must concur.

Proposed by: Daniel Zhu

Answer: $\boxed{\text{Yes}}$

Solution 1: Let $P_i(t)$ be lines in space so that $P_i(0) = A_i$, $P_i(1) = C_i$, and $P_i(2) = B_i$. Then observe that $P_iP_j^2$ are quadratics in t. The difference between any of these two is zero at $t = 0, 1, 2$, so it must be identically zero. Therefore, we find that $P_1P_2P_3P_4$ is a regular tetrahedron or a point for all t.

Now consider the signed volume $[P_1P_2P_3P_4]$. It can be written as a determinant of a matrix whose entries are constant or linear polynomials in t, so it must be $V(t)$ for some polynomial V that is at most cubic. We claim that it actually is cubic. One way to see this is that since $P_iP_j^2$ is nonconstant, it must be $\Theta(t^2)$ for large t, which implies that $|V(t)| = \Theta(t^3)$.

However, every cubic polynomial has a real root. A nondegenerate regular tetrahedron cannot have zero volume, so we conclude that P_1, P_2, P_3, P_4 are equal for some t. This yields the desired concurrency.

Solution 2: We will prove that the three tetrahedra are homothetic, which will prove the desired concurrency. Now translate so that all the tetrahedra are centered at the origin. This preserves midpoints and homothetic-ness, so it suffices to solve the problem in this case.

Let M be the matrix that takes A_i to B_i. Since the two tetrahedra are regular, we find that M is conformal and thus must be rT for some $T \in \mathrm{SO}(3)$ and $r \in \mathbb{R}$. Similarly, $\frac{1}{2}(I + M)$ must be $s'U$ for some $U \in \mathrm{SO}(3)$. Making the substitution $s = 2s'$, we must solve $I + rT = sU$.

The remainder of the solution is a straightforward computation by eigenvalues. We know that the eigenvalues of rT must have magnitude $|r|$. Also, the eigenvalues of $I + rT$, which are just one plus the eigenvalues of rT, must have magnitude $|s|$. Thus, the eigenvalues z of M must satisfy a system of the form $|z| = |r|$ and $|1 + z| = |s|$.

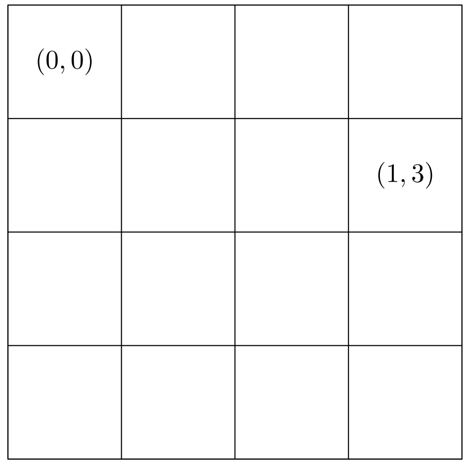
If the tetrahedra are not homothetic, then M cannot be a multiple of the identity, meaning that M must have more than one distinct eigenvalue (everything in $\mathrm{SO}(3)$ is normal and thus unitarily diagonalizable). However, by, say, geometry, that system can only have two solutions only if they are nonreal and conjugates of each other. But every 3×3 matrix has a real eigenvalue, contradiction.

5. **[12]** In an $n \times n$ square grid, n squares are marked so that every rectangle composed of exactly n grid squares contains at least one marked square. Determine all possible values of n.

Proposed by: Krit Boonsiriseth

Answer: $\boxed{n = 1, p, \text{ or } p^2 \text{ for some prime } p.}$

Solution: In this solution, we will reference cells by row and column, measured from left to right and top to bottom. The cell at $(0, 0)$ is then in the top-left corner; the cell at $(1, 3)$ is in the fourth cell of the second row.

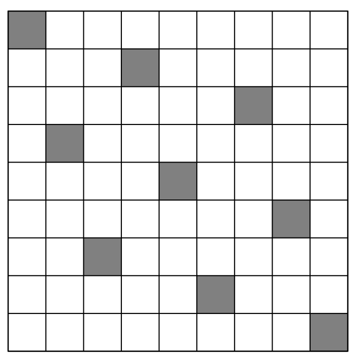


We begin by constructing solutions for the possible values of n.

For $n = 1$, we can mark the sole cell.

For $n = p$, where p is a prime, we need only ensure that each $1 \times n$ and $n \times 1$ rectangle contains a mark. This is the same as ensuring that all rows and columns contain a mark, which can be done, for example, by marking all cells (k, k) for integers $k \in [0, n)$.

For $n = p^2$, there must be a mark in each row and column and also in each $p \times p$ rectangle. To achieve this, we can mark all cells of the form $(j + kp, k + jp)$, for integers $j, k \in [0, p)$. An illustration for $n = 9$ follows:



Note that there are exactly $p^2 = n$ such cells, and that row r contains a cell in the column

$$\left\lfloor \frac{r}{p} \right\rfloor + (r \bmod p) \cdot p$$

Likewise, there is a marked cell in each column. Now, we will show that all $p \times p$ rectangles contain a marked cell. Consider the rectangle R with upper-left corner at $(xp + y, z)$ for integers $x, y \in [0, p), z \in [0, n)$. Then, if $z > x + (y - 1)p$, R contains the cell

$$\left(xp + \left\lceil \frac{z - x}{p} \right\rceil, x + \left\lceil \frac{z - x}{p} \right\rceil p \right)$$

and otherwise it contains the cell

$$\left((x + 1)p + \left\lfloor \frac{z - x}{p} \right\rfloor, x + \left\lfloor \frac{z - x}{p} \right\rfloor p + 1 \right).$$

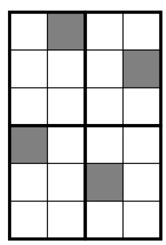
Since $z \leq n - p$, this cell is always a valid marked cell. This concludes the proof that the construction is correct.

Now, we prove that no other values of n are satisfiable. First, we again note that every row and column must have exactly one marked cell, as there are n of each and they are disjoint. Let $n = ab$ for $1 < a, b < n$ and partition the $n \times n$ grid into a $b \times a$ grid of $a \times b$ rectangles. In particular, let R_{ij} be the rectangle that has upper-left corner at (bi, aj) and lower-right corner at $(bi + b - 1, aj + a - 1)$ for $i \in [0, a), j \in [0, b)$. Then each R_{ij} must contain exactly one marked cell, since there are also n of them and they are disjoint.

Consider the set of rectangles R_{i0}. There are exactly a marked cells among these rectangles — one in each of the columns from 0 to $a - 1$, and also one in each of the a rectangles R_{i0}. Therefore, there is some permutation X of the integers $[0, a)$ so that R_{i0} has a marked cell in column X_i. Now, we will prove the claim that if a rectangle R_{ij} contains a marked cell in column c of the grid, $c \equiv X_i \pmod{a}$. We prove this by induction on j, having already demonstrated the case $j = 0$. Suppose the claim holds for some j; we will show it must also hold for $j' = j + 1$. Consider k satisfying $X_k = 0$. We know that the unique marked cell in rectangle R_{kj} is in column aj. Therefore, $R_{kj'}$ must contain a mark in column $aj + a$, as otherwise the $a \times b$ rectangle with top-left corner at $(bk, aj + 1)$ has no mark. Next, consider k' satisfying $X_k = 1$. The unique marked cell in rectangle $R_{k'j}$ is in column $aj + 1$. Also, the rectangle $R_{k'j'}$ cannot contain a marked cell in column $aj + a$, since there is already a mark in that column. So $R_{k'j'}$ must contain a marked cell in column $aj + a + 1$, as otherwise the $a \times b$ rectangle with upper-left corner at $(bk, aj + 2)$ has no mark. Proceeding in order of increasing X_i in this way forces the claim to hold for k' as well, completing the inductive proof.

Likewise, there is a permutation Y of the integers $[0, b)$ so that, if R_{ij} contains a marked cell in row r of the grid, $r \equiv Y_j \pmod{b}$.

We will now show that either X and Y are both increasing or both decreasing. Suppose not, so that for some i, j, $X_i > X_{i+1}$ and $Y_j < Y_{j+1}$ (or the same with X, Y swapped, which is handled identically). Then, consider the rectangles R_{ij}, $R_{(i+1)j}$, $R_{i(j+1)}$, and $R_{(i+1)(j+1)}$. An example for $n = 6$ follows:



Taking an $a \times b$ rectangle with upper-left corner directly below the marked cell in R_{ij}, we find that it lies entirely inside these four rectangles and yet does not contain any of their marked cells. In particular, we can note that, since Y is increasing at this point, the horizontal distance between the marks in the lower left and upper right is strictly greater than a. Also, since X is decreasing at this point, the vertical distance between the marks in the top left and lower right is strictly greater than b. This mens that an $a \times b$ rectangle with no mark exists, a contradiction. Therefore, either X is increasing or Y is decreasing.

Suppose, without loss of generality, that both are increasing (noting that the increasing and decreasing cases are simply $90°$ rotations of each other). Then the marked cell in row 1 must be $a + 1$. But if n is divisible by two distinct primes or the cube of a prime, there is no unique value for a, which is a contradiction. Therefore, n must be 1, a prime, or the square of a prime.