

.....  
**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

1. [5] The graphs of the equations

$$y = -x + 8$$

$$173y = -289x + 2021$$

on the Cartesian plane intersect at  $(a, b)$ . Find  $a + b$ .

2. [5] There are 8 lily pads in a pond numbered  $1, 2, \dots, 8$ . A frog starts on lily pad 1. During the  $i$ -th second, the frog jumps from lily pad  $i$  to  $i + 1$ , falling into the water with probability  $\frac{1}{i+1}$ . The probability that the frog lands safely on lily pad 8 without having fallen into the water at any point can be written as  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ . Find  $100m + n$ .

3. [5] Suppose

$$h \cdot a \cdot r \cdot v \cdot a \cdot r \cdot d = m \cdot i \cdot t = h \cdot m \cdot m \cdot t = 100.$$

Find  $(r \cdot a \cdot d) \cdot (t \cdot r \cdot i \cdot v \cdot i \cdot a)$ .

.....  
**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

4. [6] Find the number of ways in which the letters in “HMMTHMMT” can be rearranged so that each letter is adjacent to another copy of the same letter. For example, “MMMMTTHH” satisfies this property, but “HHTMMMMT” does not.
5. [6] A *perfect power* is an integer  $n$  that can be represented as  $a^k$  for some positive integers  $a \geq 1$  and  $k \geq 2$ . Find the sum of all prime numbers  $0 < p < 50$  such that  $p$  is 1 less than a perfect power.
6. [6] Let  $ABCD$  be a parallelogram with  $AB = 480$ ,  $AD = 200$ , and  $BD = 625$ . The angle bisector of  $\angle BAD$  meets side  $CD$  at point  $E$ . Find  $CE$ .

.....  
**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

7. [7] Two unit squares  $S_1$  and  $S_2$  have horizontal and vertical sides. Let  $x$  be the minimum distance between a point in  $S_1$  and a point in  $S_2$ , and let  $y$  be the maximum distance between a point in  $S_1$  and a point in  $S_2$ . Given that  $x = 5$ , the difference between the maximum and minimum possible values for  $y$  can be written as  $a + b\sqrt{c}$ , where  $a$ ,  $b$ , and  $c$  are integers and  $c$  is positive and square-free. Find  $100a + 10b + c$ .
8. [7] Let  $p$ ,  $q$ ,  $r$  be primes such that  $2p + 3q = 6r$ . Find  $p + q + r$ .
9. [7] Let  $n$  be an integer and

$$m = (n - 1001)(n - 2001)(n - 2002)(n - 3001)(n - 3002)(n - 3003).$$

Given that  $m$  is positive, find the minimum number of digits of  $m$ .

.....  
**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

10. [8] Squares  $ABCD$  and  $DEFG$  have side lengths 1 and  $\frac{1}{3}$ , respectively, where  $E$  is on  $\overline{CD}$  and points  $A$ ,  $D$ ,  $G$  lie on a line in that order. Line  $CF$  meets line  $AG$  at  $X$ . The length  $AX$  can be written as  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ . Find  $100m + n$ .
11. [8] Let  $n$  be a positive integer. Given that  $n^n$  has 861 positive divisors, find  $n$ .
12. [8] Alice draws three cards from a standard 52-card deck with replacement. Ace through 10 are worth 1 to 10 points respectively, and the face cards King, Queen, and Jack are each worth 10 points. The probability that the sum of the point values of the cards drawn is a multiple of 10 can be written as  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ . Find  $100m + n$ .

.....  
**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

13. [9] Find the number of ways in which the nine numbers

1, 12, 123, 1234, ..., 123456789

can be arranged in a row so that adjacent numbers are relatively prime.

14. [9] In a  $k \times k$  chessboard, a set  $S$  of 25 cells that are in a  $5 \times 5$  square is chosen uniformly at random. The probability that there are more black squares than white squares in  $S$  is 48%. Find  $k$ .
15. [9] Tetrahedron  $ABCD$  has side lengths  $AB = 6$ ,  $BD = 6\sqrt{2}$ ,  $BC = 10$ ,  $AC = 8$ ,  $CD = 10$ , and  $AD = 6$ . The distance from vertex  $A$  to face  $BCD$  can be written as  $\frac{a\sqrt{b}}{c}$ , where  $a, b, c$  are positive integers,  $b$  is square-free, and  $\gcd(a, c) = 1$ . Find  $100a + 10b + c$ .

.....  
**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

16. [10] A counter begins at 0. Then, every second, the counter either increases by 1 or resets back to 0 with equal probability. The expected value of the counter after ten seconds can be written as  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ . Find  $100m + n$ .
17. [10] Let  $ABCDEFGH$  be an equilateral octagon with  $\angle A \cong \angle C \cong \angle E \cong \angle G$  and  $\angle B \cong \angle D \cong \angle F \cong \angle H$ . If the area of  $ABCDEFGH$  is three times the area of  $ACEG$ , then  $\sin B$  can be written as  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ . Find  $100m + n$ .
18. [10] Let  $x, y, z$  be real numbers satisfying

$$\frac{1}{x} + y + z = x + \frac{1}{y} + z = x + y + \frac{1}{z} = 3.$$

The sum of all possible values of  $x + y + z$  can be written as  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ . Find  $100m + n$ .

.....  
HMMT November 2021, November 13, 2021 — GUTS ROUND

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

19. [11] Integers  $0 \leq a, b, c, d \leq 9$  satisfy

$$6a + 9b + 3c + d = 88$$

$$a - b + c - d = -6$$

$$a - 9b + 3c - d = -46$$

Find  $1000a + 100b + 10c + d$ .

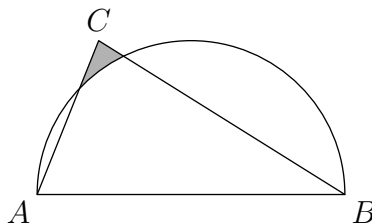
20. [11] On a chessboard, a queen attacks every square it can reach by moving from its current square along a row, column, or diagonal without passing through a different square that is occupied by a chess piece. Find the number of ways in which three indistinguishable queens can be placed on an  $8 \times 8$  chess board so that each queen attacks both others.
21. [11] Circle  $\omega$  is inscribed in rhombus  $HM_1M_2T$  so that  $\omega$  is tangent to  $\overline{HM_1}$  at  $A$ ,  $\overline{M_1M_2}$  at  $I$ ,  $\overline{M_2T}$  at  $M$ , and  $\overline{TH}$  at  $E$ . Given that the area of  $HM_1M_2T$  is 1440 and the area of  $EMT$  is 405, find the area of  $AIME$ .

.....

**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

22. [12] Two distinct squares on a  $4 \times 4$  chessboard are chosen, with each pair of squares equally likely to be chosen. A knight is placed on one of the squares. The expected value of the minimum number of moves it takes for the knight to reach the other square can be written as  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ . Find  $100m + n$ .
23. [12] Side  $\overline{AB}$  of  $\triangle ABC$  is the diameter of a semicircle, as shown below. If  $AB = 3 + \sqrt{3}$ ,  $BC = 3\sqrt{2}$ , and  $AC = 2\sqrt{3}$ , then the area of the shaded region can be written as  $\frac{a+(b+c\sqrt{d})\pi}{e}$ , where  $a, b, c, d, e$  are integers,  $e$  is positive,  $d$  is square-free, and  $\gcd(a, b, c, e) = 1$ . Find  $10000a + 1000b + 100c + 10d + e$ .



24. [12] Find the number of subsets  $S$  of  $\{1, 2, \dots, 48\}$  satisfying both of the following properties:
- For each integer  $1 \leq k \leq 24$ , exactly one of  $2k - 1$  and  $2k$  is in  $S$ .
  - There are exactly nine integers  $1 \leq m \leq 47$  so that both  $m$  and  $m + 1$  are in  $S$ .

.....  
**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

25. [13] Let  $x, y, z$  be real numbers satisfying

$$2x + y + 4xy + 6xz = -6$$

$$y + 2z + 2xy + 6yz = 4$$

$$x - z + 2xz - 4yz = -3$$

Find  $x^2 + y^2 + z^2$ .

26. [13] Let  $X$  be the number of sequences of positive integers  $a_1, a_2, \dots, a_{2047}$  that satisfy all of the following properties:

- Each  $a_i$  is either 0 or a power of 2.
- $a_i = a_{2i} + a_{2i+1}$  for  $1 \leq i \leq 1023$ .
- $a_1 = 1024$ .

Find the remainder when  $X$  is divided by 100.

27. [13]  $O$  is the center of square  $ABCD$ , and  $M$  and  $N$  are the midpoints of  $\overline{BC}$  and  $\overline{AD}$ , respectively. Points  $A', B', C', D'$  are chosen on  $\overline{AO}, \overline{BO}, \overline{CO}, \overline{DO}$ , respectively, so that  $A'B'MC'D'N$  is an equiangular hexagon. The ratio  $\frac{[A'B'MC'D'N]}{[ABCD]}$  can be written as  $\frac{a+b\sqrt{c}}{d}$ , where  $a, b, c, d$  are integers,  $d$  is positive,  $c$  is square-free, and  $\gcd(a, b, d) = 1$ . Find  $1000a + 100b + 10c + d$ .

.....  
**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

28. [15] Find the smallest positive integer  $n$  such that the divisors of  $n$  can be partitioned into three sets with equal sums.

29. [15] Kevin writes down the positive integers  $1, 2, \dots, 15$  on a blackboard. Then, he repeatedly picks two random integers  $a, b$  on the blackboard, erases them, and writes down  $\gcd(a, b)$  and  $\text{lcm}(a, b)$ . He does this until he is no longer able to change the set of numbers written on the board. Find the maximum sum of the numbers on the board after this process.

30. [15] The function  $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}$  satisfies

- $f(x, 0) = f(0, y) = 0$ , and
- $f(x, y) = f(x - 1, y) + f(x, y - 1) + x + y$

for all nonnegative integers  $x$  and  $y$ . Find  $f(6, 12)$ .

.....  
**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

31. [17] For positive integers  $n$ , let  $f(n)$  be the product of the digits of  $n$ . Find the largest positive integer  $m$  such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{m^{\lfloor \log_{10} n \rfloor}}$$

is an integer.

32. [17] There are  $N$  lockers, labeled from 1 to  $N$ , placed in clockwise order around a circular hallway. Initially, all lockers are open. Ansoon starts at the first locker and always moves clockwise. When she is at locker  $n$  and there are more than  $n$  open lockers, she keeps locker  $n$  open and closes the next  $n$  open lockers, then repeats the process with the next open locker. If she is at locker  $n$  and there are at most  $n$  lockers still open, she keeps locker  $n$  open and closes all other lockers. She continues this process until only one locker is left open. What is the smallest integer  $N > 2021$  such that the last open locker is locker 1?
33. [17] Point  $P$  lies inside equilateral triangle  $ABC$  so that  $\angle BPC = 120^\circ$  and  $AP\sqrt{2} = BP + CP$ .  $\frac{AP}{AB}$  can be written as  $\frac{a\sqrt{b}}{c}$ , where  $a, b, c$  are integers,  $c$  is positive,  $b$  is square-free, and  $\gcd(a, c) = 1$ . Find  $100a + 10b + c$ .

.....  
**HMMT November 2021, November 13, 2021 — GUTS ROUND**

Organization \_\_\_\_\_ Team \_\_\_\_\_ Team ID# \_\_\_\_\_

34. [20] Suppose two distinct competitors of the HMMT 2021 November contest are chosen uniformly at random. Let  $p$  be the probability that they can be labelled  $A$  and  $B$  so that  $A$ 's score on the General round is strictly greater than  $B$ 's, and  $B$ 's score on the theme round is strictly greater than  $A$ 's. Estimate  $P = \lfloor 10000p \rfloor$ .

An estimate of  $E$  will earn  $\left\lfloor 20 \min\left(\frac{A}{E}, \frac{E}{A}\right)^6 \right\rfloor$  points.

35. [20] The following image is 1024 pixels by 1024 pixels, and each pixel is either black or white. Let  $a$  be the proportion of pixels that are black. Estimate  $A = \lfloor 10000a \rfloor$ .

An estimate of  $E$  will earn  $\left\lfloor 20 \min\left(\frac{A}{E}, \frac{E}{A}\right)^{12} \right\rfloor$  points.

<https://i.imgur.com/W4IoGU3.png>

36. [20] Let  $N$  be the number of ways in which the letters in "HMMTHMMTHMMTHMMTHMMTHMMT" ("HMMT" repeated six times) can be rearranged so that each letter is adjacent to another copy of the same letter. For example, "MMMMMMTTTTTTHHHHHHHHHHHH" satisfies this property, but "HMMMMMTTTTTTHHHHHHHHHHHHM" does not. Estimate  $N$ .

An estimate of  $E$  will earn  $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^4 \right\rfloor$  points.