HMMT November 2021

November 13, 2021

Theme Round

1. Let n be the answer to this problem. In acute triangle ABC, point D is located on side BC so that $\angle BAD = \angle DAC$ and point E is located on AC so that $BE \perp AC$. Segments BE and AD intersect at X such that $\angle BXD = n^\circ$. Given that $\angle XBA = 16^\circ$, find the measure of $\angle BCA$.

2. Let n be the answer to this problem. An urn contains white and black balls. There are n white balls and at least two balls of each color in the urn. Two balls are randomly drawn from the urn without replacement. Find the probability, in percent, that the first ball drawn is white and the second is black.

3. Let n be the answer to this problem. Hexagon $ABCDEF$ is inscribed in a circle of radius 90. The area of $ABCDEF$ is $8n$, $AB = BC = DE = EF$, and $CD = FA$. Find the area of triangle ABC.

4. Let n be the answer to this problem. We define the digit sum of a date as the sum of its 4 digits when expressed in mmdd format (e.g. the digit sum of 13 May is $0+5+1+3 = 9$). Find the number of dates in the year 2021 with digit sum equal to the positive integer n.

5. Let n be the answer to this problem. The polynomial $x^n + ax^2 + bx + c$ has real coefficients and exactly k real roots. Find the sum of the possible values of k.

6. Let n be the answer to this problem. a and b are positive integers satisfying

$$\begin{aligned} 3a + 5b &\equiv 19 \pmod{n+1} \\ 4a + 2b &\equiv 25 \pmod{n+1} \end{aligned}$$

   Find $2a + 6b$.

7. Let n be the answer to this problem. Box B initially contains n balls, and Box A contains half as many balls as Box B. After 80 balls are moved from Box A to Box B, the ratio of balls in Box A to Box B is now $\frac{p}{q}$, where p, q are positive integers with $\gcd(p, q) = 1$. Find $100p + q$.

8. Let n be the answer to this problem. Find the number of distinct (i.e. non-congruent), non-degenerate triangles with integer side lengths and perimeter n.

9. Let n be the answer to this problem. Find the minimum number of colors needed to color the divisors of $(n - 24)!$ such that no two distinct divisors s, t of the same color satisfy $s \mid t$.

10. Let n be the answer to this problem. Suppose square $ABCD$ has side-length 3. Then, congruent non-overlapping squares $EHGF$ and $IHJK$ of side-length $\frac{n}{6}$ are drawn such that A, C, and H are collinear, E lies on BC and I lies on CD. Given that AJG is an equilateral triangle, then the area of AJG is $a + b\sqrt{c}$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.