HMMT November 2021

November 13, 2021

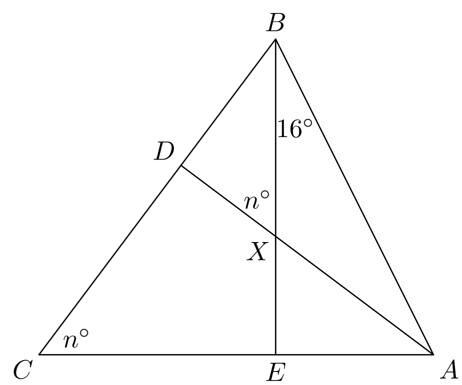
Theme Round

1. Let n be the answer to this problem. In acute triangle ABC, point D is located on side BC so that $\angle BAD = \angle DAC$ and point E is located on AC so that $BE \perp AC$. Segments BE and AD intersect at X such that $\angle BXD = n^\circ$. Given that $\angle XBA = 16^\circ$, find the measure of $\angle BCA$.

   Proposed by: Joseph Heerens

   Answer: $\boxed{53}$

   Solution:



   Since $BE \perp AC$, $\angle BAE = 90^\circ - \angle ABE = 74^\circ$. Now, $n^\circ = 180 - \angle BXA = \angle EBA + \angle BAD = 16^\circ + \frac{74^\circ}{2} = 53^\circ$.

2. Let n be the answer to this problem. An urn contains white and black balls. There are n white balls and at least two balls of each color in the urn. Two balls are randomly drawn from the urn without replacement. Find the probability, in percent, that the first ball drawn is white and the second is black.

   Proposed by: David Vulakh

   Answer: $\boxed{19}$

   Solution: Let the number of black balls in the urn be $k \geq 2$. Then the probability of drawing a white ball first is $\frac{n}{n+k}$, and the probability of drawing a black ball second is $\frac{k}{n+k-1}$. This gives us the equation

$$\frac{nk}{(n+k)(n+k-1)} = \frac{n}{100}$$

   from which we get

$$(n+k)(n+k-1) = 100k$$

   Let $m = n + k$. Since $100 \mid m(m-1)$, we must have that either 100 divides one of $m, m-1$ or 25 divides one of $m, m-1$ and 4 divides the other. Since $m, m-1 > k$, if either of m or $m-1$ is greater than or equal to 100, the product $m(m-1) > 100k$. Therefore, the only possible values for m are 25 and 76.

   If $m = 25$, we have

$$m(m-1) = 600 \implies k = 6 \implies n = 19$$

   If $m = 76$, we have

$$m(m-1) = 5700 \implies k = 57 \implies n = 19$$

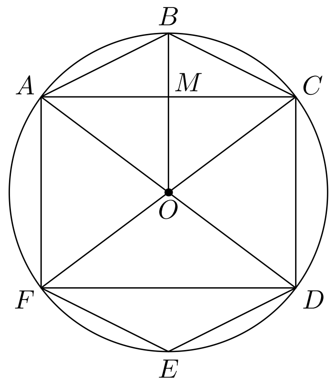
   So $n = 19$ is the unique solution.

3. Let n be the answer to this problem. Hexagon $ABCDEF$ is inscribed in a circle of radius 90. The area of $ABCDEF$ is $8n$, $AB = BC = DE = EF$, and $CD = FA$. Find the area of triangle ABC.

   Proposed by: Joseph Heerens

   Answer: $\boxed{2592}$

   Solution:



   Let O be the center of the circle, and let OB intersect AC at point M; note OB is the perpendicular bisector of AC. Since triangles ABC and DEF are congruent, $ACDF$ has area $6n$, meaning that AOC has area $3n/2$. It follows that $\frac{BM}{OM} = \frac{2}{3}$. Therefore $OM = 54$ and $MB = 36$, so by the Pythagorean theorem, $MA = \sqrt{90^2 - 54^2} = 72$. Thus, ABC has area $72 \cdot 36 = 2592$.

4. Let n be the answer to this problem. We define the digit sum of a date as the sum of its 4 digits when expressed in mmdd format (e.g. the digit sum of 13 May is 0+5+1+3 = 9). Find the number of dates in the year 2021 with digit sum equal to the positive integer n.

   Proposed by: Sheldon Kieren Tan

   Answer: $\boxed{15}$

   Solution: This problem is an exercise in how to do ugly computations efficiently.

   Let $f(n)$ be the number of days with digit sum n. Also, let $g(n)$ be the number of days with digit sum n, under the assumption that every month has 30 days. Let $h(n)$ be the number of positive integers from 1 to 30 with integer sum n. We now do computation:

n	1	2	3	4	5	6	7	8	9	10	11
$h(n)$	2	3	4	3	3	3	3	3	3	2	1

   Observe that $g(n) = \sum_{k=1}^{3} 2h(n-k) + \sum_{k=4}^{9} h(n-k)$. Also, to move from $g(n)$ to $f(n)$ we need to add in "01-31", "03-31", "05-31", "07-31", "08-31", "10-31", "12-31" and subtract "02-29", "02-30". Therefore we find

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\sum_{k=1}^{3} h(n-k)$	2	5	9	10	10	9	9	9	9	8	6	3	1						
$\sum_{k=4}^{9} h(n-k)$				2	5	9	12	15	18	19	19	18	17	15	12	9	6	3	1
$g(n)$	4	10	18	22	25	27	30	33	36	35	31	24	19	15	12	9	6	3	1
$f(n)$	4	10	18	23	25	29	30	34	36	36	32	23	19	15	12	9	6	3	1

   Evidently the answer is 15.

   While the above computation is a bit tedious in practice, the work one has to do can be dramatically cut down if one notices that if $n \leq 10$ or so, $f(n)$ is significantly larger than n. Thus one only really needs to compute the latter half of the table.

5. Let n be the answer to this problem. The polynomial $x^n + ax^2 + bx + c$ has real coefficients and exactly k real roots. Find the sum of the possible values of k.

   Proposed by: Sean Li

   Answer: $\boxed{10}$

   Solution: Note that the roots to the above polynomial must satisfy $x^n = -ax^2 - bx - c$. Therefore, it suffices to consider how many times a parabola can intersect the graph x^n. For $n \le 2$, a parabola can intersect x^n 0, 1, or 2 times, so the sum of the possible values of k is 3. Therefore, we know we must have $n > 2$. If n is odd, then a parabola can intersect x^n 0, 1, 2, or 3 times, so the sum of the possible values of k is 6, which isn't odd. Thus, we must have n is even and $n > 2$. In this case a parabola can intersect x^n 0, 1, 2, 3, or 4 times, so the sum of the possible values of k in this case is 10. Thus, we must have $n = 10$.

   Here is a more rigorous justification of the above reasoning for $n > 2$: consider $f(x) = x^n + ax^2 + bx + c$. Note that $f'''(x) = n(n-1)(n-2)x^{n-3}$, which has at most one real root. However, it is known that if a differentiable function $f(x)$ has k real roots, $f'(x)$ must have at least $k-1$ real roots, with at least one in between any pair of real roots of $f(x)$ (Proof sketch: apply Rolle's theorem many times.). Therefore, if $f(x)$ has at least five real roots, applying the above result three times yields that $f'''(x)$ has at least two real roots, a contradiction. Moreover, if $f(x)$ has four real roots and n is odd, then since nonreal roots come in pairs we know that at least one of these real roots c has multiplicity at least 2. Therefore, $f'(x)$ has three real roots in between the roots of $f(x)$, plus one real root at c. Thus $f'(x)$ has at least four real roots, implying that $f'''(x)$ has at least two real roots, again a contradiction.

6. Let n be the answer to this problem. a and b are positive integers satisfying

$$3a + 5b \equiv 19 \pmod{n+1}$$
$$4a + 2b \equiv 25 \pmod{n+1}$$

   Find $2a + 6b$.

   Proposed by: David Vulakh

   Answer: $\boxed{96}$

   Solution: Let $m = n + 1$, so that the conditions become

$$3a + 5b \equiv 19 \pmod{m} \tag{1}$$
$$4a + 2b \equiv 25 \pmod{m} \tag{2}$$
$$2a + 6b \equiv -1 \pmod{m} \tag{3}$$

   We can subtract (2) from twice (3) to obtain

$$10b \equiv -27 \pmod{m}$$

   Multiplying (1) by 2 and replacing $10b$ with -27 gives

$$6a - 27 \equiv 38 \pmod{m}$$

   So $6a \equiv 65 \pmod{m}$. Multiplying (3) by 30 and replacing $10b$ with -27 and $6a$ with 64 gives

$$650 - 486 \equiv -30 \pmod{m}$$

   Therefore $194 \equiv 0 \pmod{m}$, so $m \mid 194$. Since the prime factorization of m is $194 = 97 \cdot 2$, m must be 1, 2, 97, or 194. Condition (3) guarantees that m is odd, and $a, b > 0$ guarantees that $m = 2a + 6b + 1 \neq 1$.

So we must have $m = 97$, so $n = 96$. A valid solution is $a = 27$, $b = 7$.

7. Let n be the answer to this problem. Box B initially contains n balls, and Box A contains half as many balls as Box B. After 80 balls are moved from Box A to Box B, the ratio of balls in Box A to Box B is now $\frac{p}{q}$, where p, q are positive integers with $\gcd(p, q) = 1$. Find $100p + q$.

   Proposed by: Sheldon Kieren Tan

   Answer: $\boxed{720}$

   Solution: Originally, box A has $n/2$ balls and B has n balls. After moving, box A has $n/2 - 80$ balls and B has $n + 80$ balls. The answer to the problem is thus

$$\frac{100(n/2 - 80) + (n + 80)}{\gcd(n/2 - 80, n + 80)} = \frac{51n - 80 \cdot 99}{\gcd(n/2 - 80, n + 80)} \stackrel{?}{=} n.$$

   Write $d = \gcd(n/2-80, n+80) = \gcd(n/2-80, 240)$. Then the problem is equivalent $nd = 51n-80\cdot 99$ or $(51 - d)n = 80 \cdot 99$, with $d \mid 240$.

   Let's try to solve this. Either $51 - d$ or n must be divisible by 5. In the latter case, where n is divisible by 5, we see that d must be as well. Therefore d is either 0 or 1 mod 5.

   If n is divisible by 4, then we know that d is even and thus $51 - d$ is odd. Therefore, since $16 \mid 80 \cdot 99$, n must be divisible by 16, meaning that d is divisible by 8. Alternatively, if n is not divisible by 4, then since $16 \mid 80 \cdot 99$, $51 - d$ must be divisible by 8, meaning that d is 3 mod 8. Therefore d is either 0 or 3 mod 8.

   Putting these results together, we find that d must either be 0, 11, 16, or 35 mod 40. Since d is a divisor of 240 and less than 51, we conclude that d is either 16 or 40. If $d = 16$, then $51 - d = 35$, which does not divide $80 \cdot 99$. If $d = 40$, then we get $n = 720$, which ends up working.

8. Let n be the answer to this problem. Given $n > 0$, find the number of distinct (i.e. non-congruent), non-degenerate triangles with integer side lengths and perimeter n.

   Proposed by: Sheldon Kieren Tan

   Answer: $\boxed{48}$

   Solution: We explicitly compute the number of triangles satisfying the problem conditions for any n. There are three kinds of triangles: isosceles and scalene. (Equilateral triangles are isosceles.)

   - *Case 1: Isosceles.* A triangle with side lengths a, a, b must satisfy $2a > b$ and $2a + b = n$. So $2a$ can be any even integer in the interval $(n/2, n)$. There are thus $\lfloor (n-1)/2 \rfloor - \lfloor n/4 \rfloor$ triangles here.
   - *Case 2: Scalene.* A triangle with side lengths a, b, c must satisfy $a, b, c < n/2$ and $a + b + c = n$. There are $\binom{n-1}{2}$ triples satisfying the second condition, $3\binom{\lfloor n/2 \rfloor}{2}$ of which violate condition 1 (since at most one side is at least $n/2$, we can do casework on which side). We then remove the isosceles triangles and divide by 6 to account for ordering the sides. If there are no equilateral triangles (i.e. if $3 \nmid n$), our answer here is

$$\frac{1}{6}\left(\binom{n-1}{2} - 3\binom{\lfloor n/2 \rfloor}{2} - 3(\lfloor (n-1)/2 \rfloor - \lfloor n/4 \rfloor)\right).$$

     Otherwise, the answer is

$$\frac{1}{6}\left(\binom{n-1}{2} - 3\binom{\lfloor n/2 \rfloor}{2} - 3(\lfloor (n-1)/2 \rfloor - \lfloor n/4 \rfloor - 1) - 1\right).$$

The key observation is that almost all triangles are scalene, and there are are roughly $\frac{1}{6}(n^2/2 - 3n^2/8) = n^2/48$ scalene triangles. Hence, $n \approx 48$. Checking $n = 48$ yields

$$(23 - 12) + \frac{1}{6}\left(\binom{47}{2} - 3\binom{24}{2} - 3 \cdot (23 - 12 - 1) - 1\right) = 48.$$

Hence, the answer is 48.

Remark 1. In fact, the number of distinct triangles with perimeter n and integer sides is $[n^2/48]$ when n is even and $[(n+3)^2/48]$ when n is odd, where $[\cdot]$ is the nearest integer function. This follows by analyzing n modulo 12.

Remark 2. The problem statement originally omitted the condition $n > 0$, so an answer of 0 was also counted as correct.

9. Let n be the answer to this problem. Find the minimum number of colors needed to color the divisors of $(n-24)!$ such that no two distinct divisors s, t of the same color satisfy $s \mid t$.

Proposed by: Sean Li

Answer: $\boxed{50}$

Solution: We first answer the following question.

> Find the minimum number of colors needed to color the divisors of m such that no two distinct divisors s, t of the same color satisfy $s \mid t$.

Prime factorize $m = p_1^{e_1} \dots p_k^{e_k}$. Note that the elements

$$\begin{array}{lllll} 1, & p_1, & p_1^2, & \dots, & p_1^{e_1}, \\ & p_1^{e_1}p_2, & p_1^{e_1}p_2^2, & \dots, & p_1^{e_1}p_2^{e_2} \\ & p_1^{e_1}p_2^{e_2}p_3, & p_1^{e_1}p_2^{e_2}p_3^2, & \dots, & p_1^{e_1}p_2^{e_2}p_3^{e_3} \\ & \vdots & & & \\ & p_1^{e_1}\dots p_{k-1}^{e_{k-1}}p_k, & p_1^{e_1}\dots p_{k-1}^{e_{k-1}}p_k^2, & \dots, & p_1^{e_1}\dots p_{k-1}^{e_{k-1}}p_k^{e_k} \end{array}$$

must be pairwise different colors. Hence, we need at least $1 + e_1 + \dots + e_k$ colors. This is also sufficient: number the colors $1, 2, \dots, 1 + e_1 + \dots + e_k$, and color the divisor s with color $1 + \sum_{p \text{ prime}} \nu_p(s)$. Thus, the answer to the above question is $c(m) := 1 + e_1 + \dots + e_k$.

Now, we return to the original problem. We wish to find the integer n for which $c((n-24)!) = n$, or $c((n-24)!) - (n-24) = 24$. Let $f(k) = c(k!) - k$, so that we want to solve $f(n-24) = 24$. Note that $f(1) = 0$, while for $k > 1$ we have $f(k) - f(k-1) = c(k!) - c((k-1)!) - 1 = \Omega(k) - 1$, where $\Omega(k)$ is the number of prime factors of k with multiplicity.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\Omega(k)$		1	1	2	1	2	1	3	2	2	1	3	1	2	2	4
$f(k)$	0	0	0	1	1	2	2	4	5	6	6	8	8	9	10	13

k	17	18	19	20	21	22	23	24	25	26	27
$\Omega(k)$	1	3	1	3	2	2	1	4	2	2	3
$f(k)$	13	15	15	17	18	19	19	22	23	24	26

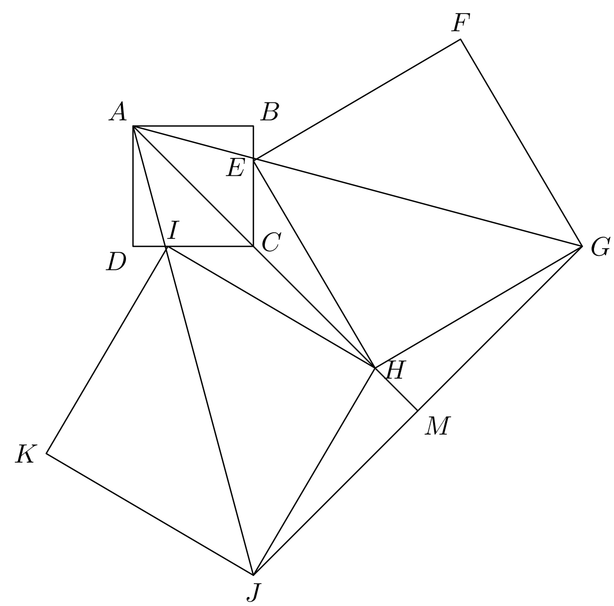
Therefore $n - 24 = 26$ and $n = 50$.

10. Let n be the answer to this problem. Suppose square $ABCD$ has side-length 3. Then, congruent non-overlapping squares $EHGF$ and $IHJK$ of side-length $\frac{n}{6}$ are drawn such that A, C, and H are collinear, E lies on BC and I lies on CD. Given that AJG is an equilateral triangle, then the area of AJG is $a + b\sqrt{c}$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.

Proposed by: Akash Das

Answer: $\boxed{48}$

Solution:



The fact that $EHGF$ and $IHJK$ have side length $n/6$ ends up being irrelevant.

Since A and H are both equidistant from G and J, we conclude that the line $ACHM$ is the perpendicular bisector of GJ.

Now, define the point C' so that the spiral similarity centered at J sends M and H to C' and I, respectively. Since $\triangle JMC' \sim \triangle JHI$, $JM \perp MC'$, so C' is on line AM. Moreover, since the spiral similarity rotates by $\angle HJI = 45^\circ$, we conclude that IC' is at a 45° angle to HM, implying that C' is on line CD. Therefore $C' = C$, implying that $\angle MJC = \angle HJI = 45^\circ$. As a result, J lies on line BC. To finish, simply note that $\angle BAJ = 75^\circ$, so by $AJ = AB/\cos 75^\circ$. So

$$[AJG] = \frac{\sqrt{3}}{4}AJ^2 = \frac{9\sqrt{3}}{4}\frac{1}{\cos^2 75^\circ} = \frac{9\sqrt{3}}{4}\frac{2}{1+\cos 150^\circ} = \frac{9\sqrt{3}}{2-\sqrt{3}} = 18\sqrt{3} + 27.$$