HMMT February 2022

February 19, 2022

Algebra and Number Theory Round

1. Positive integers a, b, and c are all powers of k for some positive integer k. It is known that the equation $ax^2 - bx + c = 0$ has exactly one real solution r, and this value r is less than 100. Compute the maximum possible value of r.

2. Compute the number of positive integers that divide at least two of the integers in the set $\{1^1, 2^2, 3^3, 4^4, 5^5, 6^6, 7^7, 8^8, 9^9, 10^{10}\}$.

3. Let $x_1, x_2, \ldots, x_{2022}$ be nonzero real numbers. Suppose that $x_k + \frac{1}{x_{k+1}} < 0$ for each $1 \leq k \leq 2022$, where $x_{2023} = x_1$. Compute the maximum possible number of integers $1 \leq n \leq 2022$ such that $x_n > 0$.

4. Compute the sum of all 2-digit prime numbers p such that there exists a prime number q for which $100q + p$ is a perfect square.

5. Given a positive integer k, let $\|k\|$ denote the absolute difference between k and the nearest perfect square. For example, $\|13\| = 3$ since the nearest perfect square to 13 is 16. Compute the smallest positive integer n such that
$$\frac{\|1\| + \|2\| + \cdots + \|n\|}{n} = 100.$$

6. Let f be a function from $\{1, 2, \ldots, 22\}$ to the positive integers such that $mn \mid f(m) + f(n)$ for all $m, n \in \{1, 2, \ldots, 22\}$. If d is the number of positive divisors of $f(20)$, compute the minimum possible value of d.

7. Let (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and (x_5, y_5) be the vertices of a regular pentagon centered at $(0, 0)$. Compute the product of all positive integers k such that the equality
$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k = y_1^k + y_2^k + y_3^k + y_4^k + y_5^k$$
must hold for all possible choices of the pentagon.

8. Positive integers $a_1, a_2, \ldots, a_7, b_1, b_2, \ldots, b_7$ satisfy $2 \leq a_i \leq 166$ and $a_i^{b_i} \equiv a_{i+1}^2 \pmod{167}$ for each $1 \leq i \leq 7$ (where $a_8 = a_1$). Compute the minimum possible value of $b_1 b_2 \cdots b_7 (b_1 + b_2 + \cdots + b_7)$.

9. Suppose $P(x)$ is a monic polynomial of degree 2023 such that
$$P(k) = k^{2023} P\left(1 - \frac{1}{k}\right)$$
for every positive integer $1 \leq k \leq 2023$. Then $P(-1) = \frac{a}{b}$, where a and b relatively prime integers. Compute the unique integer $0 \leq n < 2027$ such that $bn - a$ is divisible by the prime 2027.

10. Compute the smallest positive integer n for which there are at least two odd primes p such that
$$\sum_{k=1}^{n} (-1)^{\nu_p(k!)} < 0.$$
Note: for a prime p and a positive integer m, $\nu_p(m)$ is the exponent of the largest power of p that divides m; for example, $\nu_3(18) = 2$.