HMMT February 2022

February 19, 2022

Algebra and Number Theory Round

1. Positive integers a, b, and c are all powers of k for some positive integer k. It is known that the equation $ax^2 - bx + c = 0$ has exactly one real solution r, and this value r is less than 100. Compute the maximum possible value of r.

   Proposed by: Akash Das

   Answer: $\boxed{64}$

   Solution: Note that for there to be exactly one solution, the discriminant must be 0, so $b^2 - 4ac = 0$. Thus, b is even, so $k = 2$. Since $r = \frac{b}{2a}$, then r is also a power of 2, and the largest power of 2 less than 100 is 64. This is achieved by $(x - 64)^2 = x^2 - 128x + 4096$.

2. Compute the number of positive integers that divide at least two of the integers in the set $\{1^1, 2^2, 3^3, 4^4, 5^5, 6^6, 7^7, 8^8, 9^9, 10^{10}\}$.

   Proposed by: Daniel Zhu

   Answer: $\boxed{22}$

   Solution: For a positive integer n, let $\operatorname{rad} n$ be the product of the distinct prime factors of n. Observe that if $n \mid m^m$, all prime factors of n must divide m, so $\operatorname{rad} n \mid m$.

   Therefore, if n is such an integer, $\operatorname{rad} n$ must divide at least two of the numbers in $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, implying that $\operatorname{rad} n$ is either 1, 2, 3, or 5. These have 1, 10, 6, and 5 cases, respectively, for a total of 22.

3. Let $x_1, x_2, \ldots, x_{2022}$ be nonzero real numbers. Suppose that $x_k + \frac{1}{x_{k+1}} < 0$ for each $1 \leq k \leq 2022$, where $x_{2023} = x_1$. Compute the maximum possible number of integers $1 \leq n \leq 2022$ such that $x_n > 0$.

   Proposed by: Akash Das

   Answer: $\boxed{1010}$

   Solution: Let the answer be M. If $M > 1011$, there would exist two consecutive positive terms x_k, x_{k+1} which contradicts the assumption that $x_k + \frac{1}{x_{k+1}} < 0$. Thus, $M \leq 1011$. If $M = 1011$, then the 2022 x_is must alternate between positive and negative. WLOG, assume $x_{2k-1} > 0$ and $x_{2k} < 0$ for each k. Then, we have

$$x_{2k-1} + \frac{1}{x_{2k}} < 0 \implies |x_{2k-1}x_{2k}| < 1,$$
$$x_{2k} + \frac{1}{x_{2k+1}} < 0 \implies |x_{2k}x_{2k+1}| > 1.$$

   Multiplying the first equation over all k gives us $\prod_{i=1}^{2022} |x_i| < 1$, while multiplying the second equation over all k gives us $\prod_{i=1}^{2022} |x_i| > 1$. Thus, we must have $M < 1011$.

   $M = 1010$ is possible by the following construction:

$$1, -\frac{1}{2}, 3, -\frac{1}{4}, \ldots, 2019, -\frac{1}{2020}, -10000, -10000.$$

4. Compute the sum of all 2-digit prime numbers p such that there exists a prime number q for which $100q + p$ is a perfect square.

   Proposed by: Sheldon Kieren Tan

   Answer: $\boxed{179}$

   Solution: All squares must end with 0, 1, 4, 5, 6, or 9, meaning that p must end with 1 and 9. Moreover, since all odd squares are 1 mod 4, we know that p must be 1 mod 4. This rules all primes except for $41, 61, 29, 89$. Since $17^2 = 289, 19^2 = 361, 23^2 = 529$, 89, 61, and 29 all work. To finish, we claim that 41 does not work. If $100q + 41$ were a square, then since all odd squares are 1 mod 8 we find that $4q + 1 \equiv 1 \pmod 8$, implying that q is even. But 241 is not a square, contradiction.

   The final answer is $29 + 61 + 89 = 179$.

5. Given a positive integer k, let $\|k\|$ denote the absolute difference between k and the nearest perfect square. For example, $\|13\| = 3$ since the nearest perfect square to 13 is 16. Compute the smallest positive integer n such that

$$\frac{\|1\| + \|2\| + \cdots + \|n\|}{n} = 100.$$

   Proposed by: Carl Schildkraut

   Answer: $\boxed{89800}$

   Solution: Note that from $n = m^2$ to $n = (m+1)^2$, $\|n\|$ increases from 0 to a peak of m (which is repeated twice), and then goes back down to 0. Therefore

$$\sum_{n=1}^{m^2} \|n\| = \sum_{k=1}^{m-1} 2(1 + 2 + \cdots + k) = \sum_{k=1}^{m-1} 2\binom{k+1}{2} = 2\binom{m+1}{3} = \frac{m}{3}(m^2 - 1).$$

   In particular, if $n = m^2 - 1$,

$$\frac{\|1\| + \|2\| + \cdots + \|n\|}{n} = \frac{m}{3},$$

   so $n = 300^2 - 1$ satisfies the condition. However, this does not prove that there are not smaller solutions for n.

   Let $N = 300^2 - 1$ and suppose that $N - k$ satisfies the condition. Then, we know that

$$\frac{\|N\| + \|N-1\| + \cdots \|N-(k-1)\|}{k} = 100.$$

   Since $\|N - k\| = k + 1$ for $k \le 298$, one can show that $k = 199$ works. By looking at further terms, one can convince oneself that no larger value of k works. Thus, the answer is $300^2 - 1 - 199 = 90000 - 200 = 89800$.

6. Let f be a function from $\{1, 2, \ldots, 22\}$ to the positive integers such that $mn \mid f(m) + f(n)$ for all $m, n \in \{1, 2, \ldots, 22\}$. If d is the number of positive divisors of $f(20)$, compute the minimum possible value of d.

   Proposed by: Sheldon Kieren Tan

   Answer: $\boxed{2016}$

   Solution: Let $L = \operatorname{lcm}(1, 2, \ldots, 22)$. We claim that the possible values of $f(20)$ are the multiples of $20L$. If we can prove this, we will be done, since the minimum value of d will be the number of divisors of $20L = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, which has $7 \cdot 3^2 \cdot 2^5 = 2016$ factors.

First let's construct such an f. For any positive integer a, I claim that $f(n) = aLn$ works. Indeed, for any m, n, we find that $f(m) = aLm$ is divisible by mn, since $n \mid L$. Thus the condition is satisfied.

Now let's prove that $f(20)$ must be a multiple of $20L$. Take any prime p, and let q be the largest power of p at most 22. If $p \neq 2$, we know that $q^2 \mid 2f(q)$, meaning that $q^2 \mid f(q)$. Then, using the fact that $20q \mid f(q) + f(20)$, we find that $\gcd(20q, q^2) \mid f(q), f(q) + f(20)$, implying that

$$\nu_p(f(20)) \geq \nu_p(\gcd(20q, q^2)) = \nu_p(20q) = \nu_p(20L).$$

Now suppose $p = 2$. Then $2^8 = 16^2 \mid 2f(16)$, so $2^7 \mid f(16)$. Then, since $5 \cdot 2^6 = 20 \cdot 16 \mid f(16) + f(20)$, we find that $2^7 \mid f(20)$. Since $7 = \nu_2(20L)$, we are done.

7. Let (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and (x_5, y_5) be the vertices of a regular pentagon centered at $(0, 0)$. Compute the product of all positive integers k such that the equality

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k = y_1^k + y_2^k + y_3^k + y_4^k + y_5^k$$

must hold for all possible choices of the pentagon.

Proposed by: Daniel Zhu

Answer: $\boxed{1152}$

Solution: Without loss of generality let the vertices of the pentagon lie on the unit circle. Then, if $f(\theta) = \cos(\theta)^k$ and $g(\theta) = \sum_{j=0}^{4} f(\theta + 2j\pi/5)$, the condition becomes $g(\theta) = g(\pi/2 - \theta)$, or $g(\theta) = g(\theta + \pi/2)$, since g is an odd function.

Write $f \asymp g$ if $f = cg$ for some nonzero constant c that we don't care about. Since $\cos\theta \asymp e^{i\theta} + e^{-i\theta}$, we find that

$$f(\theta) \asymp \sum_{\ell \in \mathbb{Z}} \binom{k}{\frac{k+\ell}{2}} e^{i\ell\theta},$$

where $\binom{a}{b}$ is defined to be zero if b is not an integer in the interval $[0, a]$. It is also true that

$$\sum_{j=0}^{4} e^{i\ell(\theta + 2j\pi/5)} = \begin{cases} 5e^{i\theta} & 5 \mid \ell \\ 0 & \text{else,} \end{cases}$$

so

$$g(\theta) \asymp \sum_{\ell \in 5\mathbb{Z}} \binom{k}{\frac{k+\ell}{2}} e^{i\ell\theta}.$$

This is periodic with period $\pi/2$ if and only if all terms with ℓ not a multiple of 4 are equal to 0. However, we know that the nonzero terms are exactly the ℓ that (1) are multiples of 5, (2) are of the same parity as k, and (3) satisfy $|\ell| \leq k$. Hence, if k is even, the condition is satisfied if and only if $k < 10$ (else the $\ell = 10$ term is nonzero), and if k is odd, the condition is satisfied if and only if $k < 5$ (else the $\ell = 5$ term is nonzero). Our final answer is $1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 8 = 1152$.

8. Positive integers $a_1, a_2, \ldots, a_7, b_1, b_2, \ldots, b_7$ satisfy $2 \leq a_i \leq 166$ and $a_i^{b_i} \equiv a_{i+1}^2 \pmod{167}$ for each $1 \leq i \leq 7$ (where $a_8 = a_1$). Compute the minimum possible value of $b_1 b_2 \cdots b_7(b_1 + b_2 + \cdots + b_7)$.

Proposed by: Gregory Pylypovych

Answer: $\boxed{675}$

Solution: Let $B = b_1 b_2 \cdots b_7 - 128$. Since

$$a_1^{b_1 b_2 \cdots b_7} \equiv a_2^{2b_2 b_3 \cdots b_7} \equiv a_3^{4b_3 b_4 \cdots b_7} \equiv \cdots \equiv a_1^{128} \pmod{167},$$

we find that $a_1^B \equiv 1 \pmod{167}$. Similarly, $a_i^B \equiv 1 \pmod{167}$ for all i. Since 167 is a prime and $167 - 1 = 2 \cdot 83$, we know that the order of each individual a_i (since $a_i \neq 1$) must be either 2 of a multiple of 83. If B is not a multiple of 83, then it follows that all the a_i must be -1, which implies that all the b_i must be even, meaning that the minimum possible value of $b_1 b_2 \cdots b_7(b_1 + b_2 + \cdots b_7)$ is $2^7 \cdot 14 > 1000$.

Oh the other hand, if B is a multiple of 83, then the smallest possible values for $b_1 b_2 \cdots b_7$ are 45 and 128. If $b_1 b_2 \cdots b_7 = 45$, then the smallest possible value for $b_1 + b_2 + \cdots + b_7$ is $5+3+3+1+1+1+1 = 15$, so the minimum possible value for $b_1 b_2 \cdots b_7(b_1 + b_2 + \cdots b_7)$ is $45 \cdot 15 = 675$. This can be achieved by letting g be an element of order 83 and setting $a_1 = g, a_2 = g^{1/2}, a_3 = g^{1/4}, a_4 = g^{1/8}, a_5 = g^{1/16}, a_6 = g^{3/32}, a_7 = g^{9/64}$ (all exponents are taken mod 83).

If $b_1 b_2 \cdots b_7 \geq 128$, then by the AM-GM inequality we have

$$b_1 b_2 \cdots b_7(b_1 + b_2 + \cdots b_7) \geq 7(b_1 b_2 \cdots b_7)^{8/7} \geq 7 \cdot 2^8 > 1000.$$

Therefore 675 is optimal.

9. Suppose $P(x)$ is a monic polynomial of degree 2023 such that

$$P(k) = k^{2023} P\left(1 - \frac{1}{k}\right)$$

for every positive integer $1 \leq k \leq 2023$. Then $P(-1) = \frac{a}{b}$, where a and b relatively prime integers. Compute the unique integer $0 \leq n < 2027$ such that $bn - a$ is divisible by the prime 2027.

Proposed by: Akash Das

Answer: $\boxed{406}$

Solution: Let $n = 2023$. If $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then let

$$R(x) = x^n P(1 - \frac{1}{x}) = (x-1)^n + a_{n-1}(x-1)^n x + \cdots + a_0 x^n.$$

Then, note that $Q(x) = P(x) - R(x)$ is a polynomial of degree at most n, and it has roots $1, 2, \ldots, n$, so we have $Q(x) = k(x-1)\cdots(x-n)$ for some real constant k. Now we determine $P(x)$ in terms of $Q(x)$. If $g(x) = 1 - 1/x$, then $g(g(x)) = \frac{1}{1-x}$ and $g(g(g(x))) = x$. Therefore, we have

$$\begin{aligned}
P(x) - x^n P\left(1 - \frac{1}{x}\right) &= Q(x) \\
P\left(1 - \frac{1}{x}\right) - \left(1 - \frac{1}{x}\right)^n P\left(\frac{1}{1-x}\right) &= Q\left(1 - \frac{1}{x}\right) \\
P\left(\frac{1}{1-x}\right) - \left(\frac{1}{1-x}\right)^n P(x) &= Q\left(\frac{1}{1-x}\right).
\end{aligned}$$

Adding the first equation, x^n times the second, and $(x-1)^n$ times the third yields

$$2P(x) = Q(x) + x^n Q\left(\frac{x-1}{x}\right) + (x-1)^n Q\left(\frac{1}{1-x}\right),$$

so

$$\begin{aligned}
P(x) = \frac{k}{2}\Big((x-1)(x-2)\cdots(x-n) &+ (0x-1)(-1x-1)\cdots(-(n-1)x-1) \\
&+ (-1x+0)(-2x+1)\cdots(-nx+(n-1))\Big).
\end{aligned}$$

Therefore,

$$P(-1) = \frac{k}{2}(-(n+1)! + 0 + (2n+1)!!).$$

Also, since P is monic, we know that

$$1 = \frac{k}{2}(1 + 0 - n!),$$

so

$$P(-1) = \frac{(2n-1)!! - (n+1)!}{1 - n!}.$$

Modulo 2027, $(n+1)! = 2026!/(2026 \cdot 2025) \equiv -1/(-1 \cdot -2) \equiv -1/2$ and $n! = (n+1)!/2024 \equiv 1/6$. Also, $(2n+1)!! \equiv 0$. So our answer is

$$\frac{1/2}{1 - 1/6} = \frac{3}{5} \equiv \frac{2030}{5} = 406.$$

10. Compute the smallest positive integer n for which there are at least two odd primes p such that

$$\sum_{k=1}^{n} (-1)^{\nu_p(k!)} < 0.$$

Note: for a prime p and a positive integer m, $\nu_p(m)$ is the exponent of the largest power of p that divides m; for example, $\nu_3(18) = 2$.

Proposed by: Krit Boonsiriseth

Answer: $\boxed{229}$

Solution: Say n is p-good if $\sum_{k=1}^{n} (-1)^{\nu_p(k!)} < 0$, where p is an odd prime.

Claim. n is p-good iff

$$n + 1 = \sum_{i=0}^{k} a_i p^{2i+1},$$

where a_i is an even integer with $|a_i| < p$.

The proof of this claim will be deferred to the end of the solution as it is rather technical, and we believe that it would be more illuminating for the reader to graph the function $n \mapsto \sum_{k=1}^{n} (-1)^{\nu_p(k!)}$ and examine its properties, instead of focusing on the formal proof.

A consequence of the claim is that if n is p-good then p divides $n+1$, and $p^{2k-1} < n+1 < p^{2k}$ for some $k \in \mathbb{Z}^+$.

Now suppose n is p-good and q-good for distinct odd primes $p < q$. Then $n + 1 \geq pq > p^2$, so we must have $n + 1 > p^3$.

Checking $p = 3$, the smallest potential $n + 1$'s are

- $2 \cdot 3^3 - 2 \cdot 3 = 48$, which does not have a prime factor $q > 3$.
- $2 \cdot 3^3 = 54$, which does not have a prime factor $q > 3$.
- $2 \cdot 3^3 + 2 \cdot 3 = 60$, which does not work because 60 is the wrong size for $q = 5$.

The next value $2 \cdot 3^5 - 2 \cdot 3^3 - 2 \cdot 3$ is already bigger than 230.

Checking $p = 5$, the smallest potential $n + 1$'s are

- $2 \cdot 5^3 - 4 \cdot 5 = 230$, which works for $q = 23$.

For $p \geq 7, n+1 \geq p^3 > 230$, so 229 is the smallest value of n.

It suffices to prove the claim. We argue via a series of lemmas. We introduce the notation of $S(a,b) = \sum_{k=a}^{b-1} (-1)^{\nu_p(k!)}$. Note that n is p-good if and only if $S(0, n+1) \leq 0$.

Lemma 1. If $n \leq p^2$, $S(0,n)$ is the distance to the nearest even multiple of p.

Proof. This follows straightforwardly from the fact that $\nu_p(k!) = \lfloor k/p \rfloor$ for $n \leq p^2$.

Lemma 2. If a and b are positive integers so that $b \leq p^{\nu_p(a)}$, then $S(a, a+b) = (-1)^{\nu_p(a!)} S(0,b)$.

Proof. Note that for $0 < k < b$, $\nu_p(a+k) = \nu_p(k)$, so it follows that $\nu_p((a+k)!) = \nu_p(a!) + \nu_p(k!)$. The result follows.

Lemma 3. For any nonnegative integer a, $\nu_p((ap^2)!)$ is the same parity as $\nu_p(a!)$.

Proof. Note that $\nu_p((ap^2)!) - \nu_p(a!) = ap + a = a(p+1)$, which is even as p is odd.

Lemma 4. For a nonnegative integer a, $S(ap^2, (a+1)p^2) = p(-1)^{\nu_p(a!)}$.

Proof. Combine Lemmas 1, 2, and 3.

Lemma 5. For a nonnegative integer a, $S(0, ap^2) = pS(0,a)$.

Proof. Apply Lemma 4 and sum.

Lemma 6. If a, b are nonnegative integers with $b < p^2$, then $S(0, ap^2+b) = pS(0,a) + (-1)^{\nu_p(a!)} S(0,b)$.

Proof. Combine Lemmas 2, 3, and 5.

We are now ready to prove the claim. Call a nonnegative integer *neat* if it can be written as $\sum_{i=0}^{k} a_i p^{2i+1}$ for integers a_i with $|a_i| < p$. For a nonnegative integer n, let $P(n)$ be the following statements:

- $S(0,n) \geq 0$.
- $S(0,n) = 0$ if and only if n is neat. In this case, $\nu_p(n!)$ is even.
- $S(0,n) = 1$ if and only if $n+1$ is neat or $n-1$ is neat. If $n+1$ is neat, then $\nu_p(n!)$ is odd. If $n-1$ is neat, then $\nu_p(n!)$ is even.

It suffices to show $P(n)$ for all n, which we will prove by induction on n. The base case of $n = 0$ is obvious.

Now take some $n > 0$ and suppose $n = ap^2 + b$ for $0 \leq b < p^2$. Lemma 6 tells us that $S(0,n) = pS(0,a) + (-1)^{\nu_p(a)} S(0,b)$. Since $0 \leq S(0,b) \leq p$ (by Lemma 1), the only way for $S(0,n)$ to be less than 0 is if $S(0,a) = 0$ and $(-1)^{\nu_p(a)} = -1$, which is impossible since $P(a)$ holds.

There are two ways for $S(0,n) = 0$ to be true. The first case is that $S(0,a) = S(0,b) = 0$, which implies by $P(a)$ and Lemma 1 that a is neat and b is a multiple of $2p$. This captures the neat numbers with $a_0 \geq 0$. Note that in this case $\nu_p(n!) = \nu_p((ap^2)!) + \nu_p(b!)$ (by the same logic as Lemma 2), which is even as $\nu_p((ap^2)!)$ is even by $P(a)$ and Lemma 3 and $\nu_p(b!) = b/p$ which is even.

The second way for $S(0,n)$ to be 0 is if $S(0,a) = 1$, $S(0,b) = p$, and $(-1)^{\nu_p(a!)} = -1$. By the inductive hypothesis, this is equivalent to $a+1$ being neat and b being an odd multiple of p. This captures exactly the neat numbers with $a_0 < 0$. Also, $\nu_p(n!) = \nu_p((ap^2)!) + \nu_p(b!)$, which is even as $\nu_p(a!)$ is odd and $\nu_p(b!)$ is odd.

Analyzing the possibilities where $S(0,n) = 1$ is almost exactly the same as the above, so we will omit it here. We encourage the reader to fill in the details.