HMMT February 2022
February 19, 2022

Combinatorics Round

1. Sets A, B, and C satisfy $|A| = 92, |B| = 35, |C| = 63, |A \cap B| = 16, |A \cap C| = 51, |B \cap C| = 19$. Compute the number of possible values of $|A \cap B \cap C|$.

   Proposed by: Daniel Zhu

   Answer: $\boxed{10}$

   Solution: Suppose $|A \cap B \cap C| = n$. Then there are $16 - n$ elements in A and B but not C, $51 - n$ in A and C but not B, and $19 - n$ in B and C but not A. Furthermore, there are $25 + n$ elements that are only in A, n only in B, and $n - 7$ that are only in C. Therefore, $7 \leq n \leq 16$, so there are 10 possible values.

2. Compute the number of ways to color 3 cells in a 3×3 grid so that no two colored cells share an edge.

   Proposed by: Akash Das

   Answer: $\boxed{22}$

   Solution: If the middle square is colored, then two of the four corner squares must be colored, and there are $\binom{4}{2} = 6$ ways to do this. If the middle square is not colored, then after coloring one of the 8 other squares, there are always 6 ways to place the other two squares. However, the number of possibilities is overcounted by a factor of 3, so there are 16 ways where the middle square is not colored. This leads to a total of 22.

3. Michel starts with the string $HMMT$. An operation consists of either replacing an occurrence of H with HM, replacing an occurrence of MM with MOM, or replacing an occurrence of T with MT. For example, the two strings that can be reached after one operation are $HMMMT$ and $HMOMT$. Compute the number of distinct strings Michel can obtain after exactly 10 operations.

   Proposed by: Gabriel Wu

   Answer: $\boxed{144}$

   Solution: Each final string is of the form $HMxMT$, where x is a string of length 10 consisting of Ms and Os. Further, no two Os can be adjacent. It is not hard to prove that this is a necessary and sufficient condition for being a final string.

   Let $f(n)$ be the number of strings of length n consisting of Ms and O where no two Os are adjacent. Any such string of length $n+2$ must either end in M, in which case removing the M results in a valid string of length $n+1$, or MO, in which case removing the MO results in a valid string of length n. Therefore, $f(n+2) = f(n) + f(n+1)$. Since $f(1) = 2$ and $f(2) = 3$, applying the recursion leads to $f(10) = 144$.

4. Compute the number of nonempty subsets $S \subseteq \{-10, -9, -8, \ldots, 8, 9, 10\}$ that satisfy $|S| + \min(S) \cdot \max(S) = 0$.

   Proposed by: Akash Das

   Answer: $\boxed{335}$

   Solution: Since $\min(S) \cdot \max(S) < 0$, we must have $\min(S) = -a$ and $\max(S) = b$ for some positive integers a and b. Given a and b, there are $|S| - 2 = ab - 2$ elements left to choose, which must come from the set $\{-a+1, -a+2, \ldots, b-2, b-1\}$, which has size $a + b - 1$. Therefore the number of possibilities for a given a, b are $\binom{a+b-1}{ab-2}$.

In most cases, this binomial coefficent is zero. In particular, we must have $ab - 2 \le a + b - 1 \iff (a-1)(b-1) \le 2$. This narrows the possibilities for (a, b) to $(1, n)$ and $(n, 1)$ for positive integers $2 \le n \le 10$ (the $n = 1$ case is impossible), and three extra possibilities: $(2, 2)$, $(2, 3)$, and $(3, 2)$.

In the first case, the number of possible sets is

$$2\left(\binom{2}{0} + \binom{3}{1} + \cdots + \binom{10}{8}\right) = 2\left(\binom{2}{2} + \binom{3}{2} + \cdots + \binom{10}{2}\right) = 2\binom{11}{3} = 330.$$

In the second case the number of possible sets is

$$\binom{3}{2} + \binom{4}{4} + \binom{4}{4} = 5.$$

Thus there are 335 sets in total.

5. Five cards labeled $1, 3, 5, 7, 9$ are laid in a row in that order, forming the five-digit number 13579 when read from left to right. A swap consists of picking two distinct cards, and then swapping them. After three swaps, the cards form a new five-digit number n when read from left to right. Compute the expected value of n.

   Proposed by: Sean Li

   Answer: $\boxed{50308}$

   Solution: For a given card, let $p(n)$ denote the probability that it is in its original position after n swaps. Then $p(n+1) = p(n) \cdot \frac{3}{5} + (1 - p(n)) \cdot \frac{1}{10}$, by casework on whether the card is in the correct position or not after n swaps. In particular, $p(0) = 1$, $p(1) = 3/5$, $p(2) = 2/5$, and $p(3) = 3/10$.

   For a certain digit originally occupied with the card labeled d, we see that, at the end of the process, the card at the digit is d with probability $3/10$ and equally likely to be one of the four non-d cards with probability $7/10$. Thus the expected value of the card at this digit is

$$\frac{3d}{10} + \frac{7}{10}\frac{25-d}{4} = \frac{12d + 175 - 7d}{40} = \frac{d+35}{8}.$$

   By linearity of expectation, our final answer is therefore

$$\frac{13579 + 35 \cdot 11111}{8} = \frac{402464}{8} = 50308.$$

6. The numbers $1, 2, \ldots, 10$ are randomly arranged in a circle. Let p be the probability that for every positive integer $k < 10$, there exists an integer $k' > k$ such that there is at most one number between k and k' in the circle. If p can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, compute $100a + b$.

   Proposed by: Akash Das

   Answer: $\boxed{1390}$

   Solution: Let $n = 10$ and call two numbers *close* if there is at most one number between them and an circular permutation *focused* if only n is greater than all numbers close to it. Let A_n be the number of focused circular permutations of $\{1, 2, \ldots, n\}$.

   If $n \ge 5$, then there are 2 cases: $n-1$ is either one or two positions from n. If $n-1$ is one position from n, it is either on its left or right. In this case, one can check a permutation is focused if and only if removing n yields a focused permutation, so there are $2A_{n-1}$ permutations in this case. If $n-1$ is two positions from n, there are $n-2$ choices for k, the element that lies between n and $n-1$. One can show that this permutation is focused if and only if removing both n and k and relabeling the numbers yields a focused permutation, so there are $2(n-2)A_{n-2}$ permutations in this case. Thus, we have $A_n = 2A_{n-1} + 2(n-2)A_{n-2}$.

If we let $p_n = A_n/(n-1)!$ the probability that a random circular permutation is focused, then this becomes

$$p_n = \frac{2p_{n-1} + 2p_{n-2}}{n-1}.$$

Since $p_3 = p_4 = 1$, we may now use this recursion to calculate

$$p_5 = 1, p_6 = \frac{4}{5}, p_7 = \frac{3}{5}, p_8 = \frac{2}{5}, p_9 = \frac{1}{4}, p_{10} = \frac{13}{90}.$$

7. Let $S = \{(x,y) \in \mathbb{Z}^2 \mid 0 \le x \le 11, 0 \le y \le 9\}$. Compute the number of sequences $(s_0, s_1, \ldots, s_n)$ of elements in S (for any positive integer $n \ge 2$) that satisfy the following conditions:

- $s_0 = (0,0)$ and $s_1 = (1,0)$,
- $s_0, s_1, \ldots, s_n$ are distinct,
- for all integers $2 \le i \le n$, s_i is obtained by rotating s_{i-2} about s_{i-1} by either 90° or 180° in the clockwise direction.

Proposed by: Gabriel Wu

Answer: $\boxed{646634}$

Solution: Let a_n be the number of such possibilities where there n 90° turns. Note that $a_0 = 10$ and $a_1 = 11 \cdot 9$.

Now suppose $n = 2k$ with $k \ge 1$. The path traced out by the s_i is uniquely determined by a choice of $k+1$ nonnegative x-coordinates and k positive y-coordinates indicating where to turn and when to stop. If $n = 2k+1$, the path is uniquely determined by a choice of $k+1$ nonnegative x-coordinates and $k+1$ positive y-coordinates.

As a result, our final answer is

$$10 + 11 \cdot 9 + \binom{12}{2}\binom{9}{1} + \binom{12}{2}\binom{9}{2} + \cdots = -12 + \binom{12}{0}\binom{9}{0} + \binom{12}{1}\binom{9}{0} + \binom{12}{1}\binom{9}{1} + \cdots.$$

One can check that

$$\sum_{k=0}^{9}\binom{12}{k}\binom{9}{k} = \sum_{k=0}^{9}\binom{12}{k}\binom{9}{9-k} = \binom{21}{9}$$

by Vandermonde's identity. Similarly,

$$\sum_{k=0}^{9}\binom{12}{k+1}\binom{9}{k} = \sum_{k=0}^{9}\binom{12}{k+1}\binom{9}{9-k} = \binom{21}{10}.$$

Thus our final answer is

$$\begin{aligned}\binom{22}{10} - 12 &= -12 + \frac{22 \cdot 21 \cdot 2 \cdot 19 \cdot 2 \cdot 17 \cdot 2 \cdot 15 \cdot 2 \cdot 13}{6!} \\ &= -12 + 7 \cdot 11 \cdot 13 \cdot \frac{2^5 \cdot 3^2 \cdot 5 \cdot 19 \cdot 17}{2^4 \cdot 3^2 \cdot 5} \\ &= -12 + 1001 \cdot 2 \cdot 17 \cdot 19 \\ &= 646646 - 12 = 646634.\end{aligned}$$

8. Random sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are chosen so that every element in each sequence is chosen independently and uniformly from the set $\{0, 1, 2, 3, \ldots, 100\}$. Compute the expected value of the smallest nonnegative integer s such that there exist positive integers m and n with

$$s = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

Proposed by: Gabriel Wu

Answer: $\boxed{2550}$

Solution: Let's first solve the problem, ignoring the possibility that the a_i and b_i can be zero. Call a positive integer s an *A-sum* if $s = \sum_{i=1}^{m} a_i$ for some nonnegative integer m (in particular, 0 is always an A-sum). Define the term *B-sum* similarly. Let E be the expected value of the smallest positive integer that is both an A-sum and a B-sum.

The first key observation to make is that if s is both an A-sum and a B-sum, then the distance to the next number that is both an A-sum and a B-sum is E. To see this, note that if

$$s = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_i$$

the distance to the next number that is both an A-sum and a B-sum is the minimal positive integer t so that there exist m' and n' so that

$$t = \sum_{i=1}^{m'} a_{m+i} = \sum_{j=1}^{n'} b_{n+i}.$$

This is the same question of which we defined E to be the answer, but with renamed variables, so the expected value of t is E. As a result, we conclude that the expected density of numbers that are both A-sums and B-sums is $\frac{1}{E}$.

We now compute this density. Note that since the expected value of a_i is $\frac{101}{2}$, the density of A-sums is $\frac{2}{101}$. Also, the density of B-sums if $\frac{2}{101}$. Moreover, as n goes to infinity, the probability that n is an A-sum approaches $\frac{2}{101}$ and the probability that n is a B-sum approaches $\frac{2}{101}$. Thus, the density of numbers that are simultaneously A-sums and B-sums is $\frac{4}{101^2}$, so $E = \frac{101^2}{4}$.

We now add back the possibility that some of the a_i and b_i can be 0. The only way this changes our answer is that the s we seek can be 0, which happens if and only if $a_1 = b_1 = 0$. Thus our final answer is

$$\frac{1}{101^2} \cdot 0 + \frac{101^2 - 1}{101^2} \cdot \frac{101^2}{4} = \frac{101^2 - 1}{4} = 2550.$$

9. Consider permutations $(a_0, a_1, \ldots, a_{2022})$ of $(0, 1, \ldots, 2022)$ such that

- $a_{2022} = 625$,
- for each $0 \le i \le 2022$, $a_i \ge \frac{625i}{2022}$,
- for each $0 \le i \le 2022$, $\{a_i, \ldots, a_{2022}\}$ is a set of consecutive integers (in some order).

The number of such permutations can be written as $\frac{a!}{b!c!}$ for positive integers a, b, c, where $b > c$ and a is minimal. Compute $100a + 10b + c$.

Proposed by: Milan Haiman

Answer: $\boxed{216695}$

Solution: Ignore the second condition for now. The permutations we seek are in bijection with the $\binom{2022}{625}$ ways to choose 625 indices $i \le 2021$ so that $a_i < 625$. These are in bijection with up-right lattice paths from $(0,0)$ to $(625, 1397)$ in the following way: a step $(i,j) \to (i+1,j)$ indicates that $a_{i+j} = i$, while a step $(i,j) \to (i,j+1)$ indicates that $a_{i+j} = 2022 - j$.

Under this bijection, the second condition now becomes: for every right step $(i,j) \to (i+1,j)$, we have $i \ge \frac{625}{2022}(i+j)$, which is equivalent to $j \le \frac{1397}{625} i$. In other words, we want to count the number of paths from $(0,0)$ to $(625, 1397)$ that stay under the line $y = \frac{1397}{625}x$.

This can be counted via a standard shifting argument. Given a path from $(0,0)$ to $(625, 1397)$, one can shift it by moving the first step to the end. We claim that exactly one of these cyclic shifts has the property of lying under the lines $y = \frac{1397}{625}x$. If we can show this, it follows that the answer is $\frac{2021!}{1397!625!}$, since, as $\gcd(2022, 625) = 1$, all the cyclic shifts are distinct. (It is true that the 2021 is minimal and that, given the numerator, the form of the denominator is unique. However, proving this is a bit annoying so we omit it here.)

To see that exactly one cyclic shift lies under the line, imagine extending a path infinitely in both directions in a periodic manner. A cyclic shift corresponds to taking a subset of this path between two points P and Q at distance 2022 along the path. Note that the condition of the path lying below the line corresponds to the infinite path lying below line PQ, so the unique P, Q that satisfy this condition are those that lie on the highest line of slope $\frac{1397}{625}$ that touches the path. Since $\gcd(2022, 625) = 1$, these points are unique.

10. Let S be a set of size 11. A random 12-tuple $(s_1, s_2, \ldots, s_{12})$ of elements of S is chosen uniformly at random. Moreover, let $\pi\colon S \to S$ be a permutation of S chosen uniformly at random. The probability that $s_{i+1} \ne \pi(s_i)$ for all $1 \le i \le 12$ (where $s_{13} = s_1$) can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Compute a.

Proposed by: Daniel Zhu

Answer: $\boxed{1000000000004}$

Solution: Given a permutation π, let $\nu(\pi)$ be the number of fixed points of π. We claim that if we fix π, then the probability that the condition holds, over the randomness of s_i, is $\frac{10^{12}+\nu(\pi^{12})-1}{11^{12}}$. Note that a point in S is a fixed point of π^{12} if and only if the length of its cycle in π is 1, 2, 3, 4, or 6, which happens with probability $\frac{5}{11}$, as each cycle length from 1 to 11 is equally likely. Therefore, the answer is

$$\mathbb{E}_\pi\left[\frac{10^{12}+\nu(\pi^{12})-1}{11^{12}}\right] = \frac{10^{12}+4}{11^{12}}.$$

Since 11 does not divide $10^{12}+4$ this fraction is simplified.

We now prove the claim. Instead of counting $(s_1, s_2, \ldots, s_{12})$, we count tuples $(t_1, t_2, \ldots, t_{12})$ so that $t_i \ne t_{i+1}$ for $1 \le i \le 11$ and $t_1 \ne \pi^{12}(t_{12})$. A bijection between the two is to let $t_i = \pi^{-i}(s_i)$. To do this, fix a t_1. If t_1 is a fixed point of π^{12}, we need to count the possibilities for $t_2, \ldots, t_{12}$ so that $t_1 \ne t_2, t_2 \ne t_3, \ldots, t_{12} \ne t_1$. This can be done via recursion: if a_k is the number of $t_2, \ldots, t_{k+1}$ so that $t_1 \ne t_2, t_2 \ne t_3, \ldots, t_{k+1} \ne t_1$, then $a_0 = 0$, while for $n \ge 0$ we have $a_{n+1} = 9a_n + 10(10^n - a_n) = 10^{n+1} - a_n$; thus $a_{11} = 10^{11} - 10^{10} + \cdots + 10^1 = \frac{1}{11}(10^{12}+10)$. Similarly, if t_1 is not a fixed point of π^{12}, there are $\frac{1}{11}(10^{12}-1)$ ways. Therefore, number of possible $(t_1, \ldots, t_n)$ is

$$\frac{10^{12}}{11}(\nu(\pi^{12}) + (11 - \nu(\pi^{12}))) + \frac{1}{11}(10\nu(\pi^{12}) - (11 - \nu(\pi^{12}))) = 10^{12} + \nu(\pi^{12}) - 1,$$

as desired.