HMMT February 2022

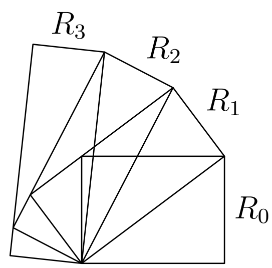
February 19, 2022

Geometry Round

1. Let ABC be a triangle with $\angle A = 60^\circ$. Line ℓ intersects segments AB and AC and splits triangle ABC into an equilateral triangle and a quadrilateral. Let X and Y be on ℓ such that lines BX and CY are perpendicular to ℓ. Given that $AB = 20$ and $AC = 22$, compute XY.

2. Rectangle R_0 has sides of lengths 3 and 4. Rectangles R_1, R_2, and R_3 are formed such that:

   - all four rectangles share a common vertex P,
   - for each $n = 1, 2, 3$, one side of R_n is a diagonal of R_{n-1},
   - for each $n = 1, 2, 3$, the opposite side of R_n passes through a vertex of R_{n-1} such that the center of R_n is located counterclockwise of the center of R_{n-1} with respect to P.

   Compute the total area covered by the union of the four rectangles.

3. Let $ABCD$ and $AEFG$ be unit squares such that the area of their intersection is $\frac{20}{21}$. Given that $\angle BAE < 45^\circ$, $\tan \angle BAE$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

4. Parallel lines $\ell_1, \ell_2, \ell_3, \ell_4$ are evenly spaced in the plane, in that order. Square $ABCD$ has the property that A lies on ℓ_1 and C lies on ℓ_4. Let P be a uniformly random point in the interior of $ABCD$ and let Q be a uniformly random point on the perimeter of $ABCD$. Given that the probability that P lies between ℓ_2 and ℓ_3 is $\frac{53}{100}$, the probability that Q lies between ℓ_2 and ℓ_3 can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

5. Let triangle ABC be such that $AB = AC = 22$ and $BC = 11$. Point D is chosen in the interior of the triangle such that $AD = 19$ and $\angle ABD + \angle ACD = 90^\circ$. The value of $BD^2 + CD^2$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

6. Let $ABCD$ be a rectangle inscribed in circle Γ, and let P be a point on minor arc AB of Γ. Suppose that $PA \cdot PB = 2$, $PC \cdot PD = 18$, and $PB \cdot PC = 9$. The area of rectangle $ABCD$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers and b is a squarefree positive integer. Compute $100a + 10b + c$.

7. Point P is located inside a square $ABCD$ of side length 10. Let O_1, O_2, O_3, O_4 be the circumcenters of PAB, PBC, PCD, and PDA, respectively. Given that $PA + PB + PC + PD = 23\sqrt{2}$ and the area of $O_1O_2O_3O_4$ is 50, the second largest of the lengths $O_1O_2, O_2O_3, O_3O_4, O_4O_1$ can be written as $\sqrt{\frac{a}{b}}$, where a and b are relatively prime positive integers. Compute $100a + b$.

8. Let $\mathcal{E}$ be an ellipse with foci A and B. Suppose there exists a parabola $\mathcal{P}$ such that

   - $\mathcal{P}$ passes through A and B,
   - the focus F of $\mathcal{P}$ lies on $\mathcal{E}$,
   - the orthocenter H of $\triangle FAB$ lies on the directrix of $\mathcal{P}$.

If the major and minor axes of $\mathcal{E}$ have lengths 50 and 14, respectively, compute $AH^2 + BH^2$.

9. Let $A_1B_1C_1$, $A_2B_2C_2$, and $A_3B_3C_3$ be three triangles in the plane. For $1 \le i \le 3$, let D_i, E_i, and F_i be the midpoints of B_iC_i, A_iC_i, and A_iB_i, respectively. Furthermore, for $1 \le i \le 3$ let G_i be the centroid of $A_iB_iC_i$.

   Suppose that the areas of the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$, $D_1D_2D_3$, $E_1E_2E_3$, and $F_1F_2F_3$ are 2, 3, 4, 20, 21, and 2020, respectively. Compute the largest possible area of $G_1G_2G_3$.

10. Suppose ω is a circle centered at O with radius 8. Let AC and BD be perpendicular chords of ω. Let P be a point inside quadrilateral $ABCD$ such that the circumcircles of triangles ABP and CDP are tangent, and the circumcircles of triangles ADP and BCP are tangent. If $AC = 2\sqrt{61}$ and $BD = 6\sqrt{7}$, then OP can be expressed as $\sqrt{a} - \sqrt{b}$ for positive integers a and b. Compute $100a + b$.