HMMT February 2022

February 19, 2022

Geometry Round

1. Let ABC be a triangle with $\angle A = 60^\circ$. Line ℓ intersects segments AB and AC and splits triangle ABC into an equilateral triangle and a quadrilateral. Let X and Y be on ℓ such that lines BX and CY are perpendicular to ℓ. Given that $AB = 20$ and $AC = 22$, compute XY.

   Proposed by: Akash Das

   Answer: $\boxed{21}$

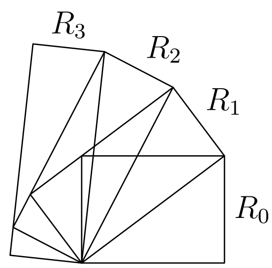
   Solution: Let the intersection points of ℓ with AB and AC be B' and C'. Note that $AB' + AC' = 2B'C'$, $BB' = 2XB'$, and $CC' = 2YC'$. Adding gives us

$$AB + AC = AB' + AC' + BB' + CC' = 2(B'C' + XB' + YC') = 2XY.$$

   Thus, $XY = \frac{20+22}{2} = 21$.

2. Rectangle R_0 has sides of lengths 3 and 4. Rectangles R_1, R_2, and R_3 are formed such that:

   - all four rectangles share a common vertex P,
   - for each $n = 1, 2, 3$, one side of R_n is a diagonal of R_{n-1},
   - for each $n = 1, 2, 3$, the opposite side of R_n passes through a vertex of R_{n-1} such that the center of R_n is located counterclockwise of the center of R_{n-1} with respect to P.



   Compute the total area covered by the union of the four rectangles.

   Proposed by: Grace Tian

   Answer: $\boxed{30}$

   Solution: Let $ABCD$ be R_0 such that $\overline{AB} = 3$ and $\overline{BC} = 4$. Then, let $\overline{AC}$ be a side length of R_1 and let the other two vertices be E and F such that B lies on segment EF. Notice that the area of $\triangle ABC$ is both half of the area of R_0 and half of the area of R_1. This means forming R_1 adds half of the area of R_0 to the union of rectangles. Similarly, forming R_2 adds half of the area of R_1 to the union of all rectangles, and the same for R_3. This means the total area of the union of rectangles is given by

$$[R_0] + \frac{1}{2}[R_1] + \frac{1}{2}[R_2] + \frac{1}{2}[R_3] = [R_0] + \frac{1}{2}[R_0] + \frac{1}{2}[R_0] + \frac{1}{2}[R_0] = \frac{5}{2}[R_0] = \frac{5}{2}(3 \cdot 4) = 30.$$

   Note that in the above equation, $[X]$ denotes the area of shape X.

3. Let $ABCD$ and $AEFG$ be unit squares such that the area of their intersection is $\frac{20}{21}$. Given that $\angle BAE < 45^\circ$, $\tan \angle BAE$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $100a + b$.

Proposed by: Benjamin Shimabukuro

Answer: $\boxed{4940}$

Solution: Suppose the two squares intersect at a point $X \neq A$. If $\mathcal{S}$ is the region formed by the intersection of the squares, note that line AX splits $\mathcal{S}$ into two congruent pieces of area $\frac{10}{21}$. Each of these pieces is a right triangle with one leg of length 1, so the other leg must have length $\frac{20}{21}$. Thus, if the two squares are displaced by an angle of θ, then $90 - \theta = 2\arctan\frac{20}{21}$. Though there is some ambiguity in how the points are labeled, the fact that $\angle BAF < 45°$ tells us that $\angle BAF = \theta$. Therefore

$$\tan\angle BAF = \frac{1}{\tan(2\arctan\frac{20}{21})} = \frac{1-\frac{20^2}{21^2}}{2\cdot\frac{20}{21}} = \frac{41}{840}.$$

4. Parallel lines $\ell_1, \ell_2, \ell_3, \ell_4$ are evenly spaced in the plane, in that order. Square $ABCD$ has the property that A lies on ℓ_1 and C lies on ℓ_4. Let P be a uniformly random point in the interior of $ABCD$ and let Q be a uniformly random point on the perimeter of $ABCD$. Given that the probability that P lies between ℓ_2 and ℓ_3 is $\frac{53}{100}$, the probability that Q lies between ℓ_2 and ℓ_3 can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

   Proposed by: Daniel Zhu

   Answer: $\boxed{6100}$

   Solution: The first thing to note is that the area of $ABCD$ does not matter in this problem, so for the sake of convenience, introduce coordinates so that $A = (0, 0)$, $B = (1, 0)$, and $C = (0, 1)$.

   Suppose A and B lie on the same side of ℓ_2. Then, by symmetry, C and D lie on the same side of ℓ_3. Now suppose BC intersects ℓ_2 and ℓ_3 at X and Y, respectively, and that DA intersects ℓ_2 and ℓ_3 at U and V, respectively. Note that $XYVU$ is a parallelogram. Since $BC = BX + XY + YC = BX + 2XY > 2XY$, we have that XY is less than half the side length of the square, so the area of $XYVU$ is at most half of the area of square $ABCD$. However, since $0.53 > \frac{1}{2}$, this can't happen. Similar reasoning applies if B and C lie on the same side of ℓ_3. Therefore, points B and D lie between ℓ_2 and ℓ_3.

   Let AB and AD intersect ℓ_2 at points M and N, respectively. Let $r = AM$ and $s = AN$. By symmetry, $[AMN] = 0.235$, so $rs = 0.47$. Additionally, in coordinates line ℓ_2 is just $\frac{x}{r} + \frac{y}{s} = 1$. Therefore line ℓ_4 is given by $\frac{x}{r} + \frac{y}{s} = 3$. Since $C = (1, 1)$ lies on this line, $\frac{1}{r} + \frac{1}{s} = 3$.

   The answer that we want is
$$1 - \frac{2r+2s}{4} = 1 - \frac{r+s}{2}.$$

   On the other hand, the condition $\frac{1}{r} + \frac{1}{s} = 3$ rearranges to $3rs = r + s$, so $r + s = 1.41$. Thus the answer is $1 - \frac{1.41}{2} = 0.295 = \frac{59}{200}$.

5. Let triangle ABC be such that $AB = AC = 22$ and $BC = 11$. Point D is chosen in the interior of the triangle such that $AD = 19$ and $\angle ABD + \angle ACD = 90°$. The value of $BD^2 + CD^2$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $100a + b$.

   Proposed by: Akash Das

   Answer: $\boxed{36104}$

   Solution: Rotate triangle ABD about A so that B coincides with C. Let D map to D' under this. Note that CDD' is a right triangle with right angle at C. Also, note that ADD' is similar to ABC. Thus, we have $DD' = \frac{AD}{2} = \frac{19}{2}$. Finally, note that

$$BD^2 + CD^2 = CD'^2 + CD^2 = DD'^2 = \frac{361}{4}.$$

6. Let $ABCD$ be a rectangle inscribed in circle Γ, and let P be a point on minor arc AB of Γ. Suppose that $PA \cdot PB = 2$, $PC \cdot PD = 18$, and $PB \cdot PC = 9$. The area of rectangle $ABCD$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers and b is a squarefree positive integer. Compute $100a + 10b + c$.

   Proposed by: Ankit Bisain

   Answer: $\boxed{21055}$

   Solution: We have
$$PD \cdot PA = \frac{(PA \cdot PB)(PD \cdot PC)}{(PB \cdot PC)} = \frac{2 \cdot 18}{9} = 4.$$
   Let $\alpha = \angle DPC = 180^\circ - \angle APB$ and $\beta = \angle APD = \angle BPC$. Note that $\alpha + \beta = 90^\circ$. We have, letting $x = AB = CD$ and $y = AD = BC$,
$$2[PAD] + 2[PBC] = y(d(P, AD) + d(P, BC)) = y \cdot x = [ABCD].$$
   Here $d(X, \ell)$ is used to denote the distance from X to line ℓ. By the trig area formula, the left-hand side is
$$PA \cdot PD \cdot \sin\beta + PB \cdot PC \cdot \sin\beta = 13 \sin\beta.$$
   Similarly, we have $[ABCD] = 16 \sin\alpha$. Thus, letting $K = [ABCD]$,
$$1 = \sin^2\alpha + \sin^2\beta = \frac{K^2}{13^2} + \frac{K^2}{16^2} = \frac{425}{13^2 \cdot 16^2} K^2$$
   giving $K = \frac{208}{\sqrt{425}} = \frac{208\sqrt{17}}{85}$.

7. Point P is located inside a square $ABCD$ of side length 10. Let O_1, O_2, O_3, O_4 be the circumcenters of PAB, PBC, PCD, and PDA, respectively. Given that $PA + PB + PC + PD = 23\sqrt{2}$ and the area of $O_1O_2O_3O_4$ is 50, the second largest of the lengths $O_1O_2, O_2O_3, O_3O_4, O_4O_1$ can be written as $\sqrt{\frac{a}{b}}$, where a and b are relatively prime positive integers. Compute $100a + b$.

   Proposed by: Daniel Zhu

   Answer: $\boxed{16902}$

   Solution: Note that O_1O_3 and O_2O_4 are perpendicular and intersect at O, the center of square $ABCD$. Also note that $O_1O_2, O_2O_3, O_3O_4, O_4O_1$ are the perpendiculars of PB, PC, PD, PA, respectively. Let $d_1 = OO_1$, $d_2 = OO_2$, $d_3 = OO_3$, and $d_4 = OO_4$. Note that that since the area of $O_1O_2O_3O_4 = 50$, we have that $(d_1 + d_3)(d_2 + d_4) = 100$. Also note that the area of octagon $AO_1BO_2CO_3DO_4$ is twice the area of $O_1O_2O_3O_4$, which is the same as the area of $ABCD$. Note that the difference between the area of this octagon and $ABCD$ is $\frac{1}{2} \cdot 10[(d_1 - 5) + (d_2 - 5) + (d_3 - 5) + (d_4 - 5)]$. Since this must equal 0, we have $d_1 + d_2 + d_3 + d_4 = 20$. Combining this with the fact that $(d_1 + d_3)(d_2 + d_4) = 100$ gives us $d_1 + d_3 = d_2 + d_4 = 10$, so $O_1O_3 = O_2O_4 = 10$. Note that if we translate AB by 10 to coincide with DC, then O_1 would coincide with O_3, and thus if P translates to P', then $PCP'D$ is cyclic. In other words, we have $\angle APB$ and $\angle CPD$ are supplementary.

   Fix any $\alpha \in (0^\circ, 180^\circ)$. There are at most two points P in $ABCD$ such that $\angle APB = \alpha$ and $\angle CPD = 180^\circ - \alpha$ (two circular arcs intersect at most twice). Let P' denote the unique point on AC such that $\angle AP'B = \alpha$, and let P^* denote the unique point on BD such that $\angle AP^*B = \alpha$. Note that it is not hard to see that in we have $\angle CP'D = \angle CP^*D = 180^\circ - \alpha$. Thus, we have $P = P'$ or $P = P^*$, so P must lie on one of the diagonals. Without loss of generality, assume $P = P'$ (P is on AC). Note that $O_1O_2O_3O_4$ is an isosceles trapezoid with bases O_1O_4 and O_2O_3. Additionally, the height of the trapezoid is $\frac{AC}{2} = 5\sqrt{2}$. Since the area of trapezoid is $O_1O_2O_3O_4$, we have the midlength of the trapezoid is $\frac{50}{5\sqrt{2}} = 5\sqrt{2}$. Additionally, note that $\angle PO_1B = 2\angle PAB = 90°$. Similarly $\angle PO_2B = 90°$. Combining this with the fact that O_1O_2 perpendicular bisects PB, we get that PO_1BO_2 is a square, so $O_1O_2 = PB = \frac{23\sqrt{2}-10\sqrt{2}}{2} = \frac{13\sqrt{2}}{2} = \sqrt{\frac{169}{2}}$. Since this is the second largest side of $O_1O_2O_3O_4$, we are done.

8. Let $\mathcal{E}$ be an ellipse with foci A and B. Suppose there exists a parabola $\mathcal{P}$ such that

   - $\mathcal{P}$ passes through A and B,
   - the focus F of $\mathcal{P}$ lies on $\mathcal{E}$,
   - the orthocenter H of $\triangle FAB$ lies on the directrix of $\mathcal{P}$.

   If the major and minor axes of $\mathcal{E}$ have lengths 50 and 14, respectively, compute $AH^2 + BH^2$.

   Proposed by: Jeffrey Lu

   Answer: $\boxed{2402}$

   Solution: Let D and E be the projections of A and B onto the directrix of $\mathcal{P}$, respectively. Also, let ω_A be the circle centered at A with radius $AD = AF$, and define ω_B similarly.

   If M is the midpoint of $\overline{DE}$, then M lies on the radical axis of ω_A and ω_B since $MD^2 = ME^2$. Since F lies on both ω_A and ω_B, it follows that MF is the radical axis of the two circles. Moreover, $MF \perp AB$, so we must have $M = H$.

   Let N be the midpoint of $\overline{AB}$. We compute that $AD+BE = AF+FB = 50$, so $HN = \frac{1}{2}(AD+BE) = 25$. Since $AB = 2\sqrt{25^2-7^2} = 48$, we have

$$\begin{aligned} 25^2 = HN^2 &= \tfrac{1}{2}\left(AH^2 + BH^2\right) - \tfrac{1}{4}AB^2 \\ &= \tfrac{1}{2}\left(AH^2 + BH^2\right) - 24^2. \end{aligned}$$

   by the median lengtb formula. Thus $AH^2 + BH^2 = 2(25^2 + 24^2) = 2402$.

9. Let $A_1B_1C_1$, $A_2B_2C_2$, and $A_3B_3C_3$ be three triangles in the plane. For $1 \le i \le 3$, let D_i, E_i, and F_i be the midpoints of B_iC_i, A_iC_i, and A_iB_i, respectively. Furthermore, for $1 \le i \le 3$ let G_i be the centroid of $A_iB_iC_i$.

   Suppose that the areas of the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$, $D_1D_2D_3$, $E_1E_2E_3$, and $F_1F_2F_3$ are 2, 3, 4, 20, 21, and 2020, respectively. Compute the largest possible area of $G_1G_2G_3$.

   Proposed by: Daniel Zhu

   Answer: $\boxed{917}$

   Solution: Let $P_i(x,y,z)$ be the point with barycentric coordinates (x,y,z) in triangle $A_iB_iC_i$. Note that since this is linear in x,y, and z, the signed area of triangle $P_1(x,y,z)P_2(x,y,z)P_3(x,y,z)$ is a homogenous quadratic polynomial in x, y, and z; call it $f(x,y,z)$.

   We now claim that

$$f(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}) = \frac{4f(\frac{1}{2},\frac{1}{2},0) + 4f(\frac{1}{2},0,\frac{1}{2}) + 4f(0,\frac{1}{2},\frac{1}{2}) - f(1,0,0) - f(0,1,0) - f(0,0,1)}{9}.$$

   This is easy to verify for $f \in \{x^2, y^2, z^2, xy, xz, yz\}$, after which the statement follows for general f by linearity. Then, assuming that we can arbitrarily choose the signs of the areas, the area is maximized at

$$\frac{4 \cdot 2061 + 9}{9} = 229 \cdot 4 + 1 = 917.$$

   Now it remains to show that this best-case scenario is actually possible. The first step is to first show that these values from an actual f, i.e. that one can fit a homogenous quadratic polynomial through every six possible values for f at the six given points. One way to see this is to note that by choosing the coefficients for x^2, y^2, and z^2, the values at the vertices of the triangle can be matched, while adding any of the xy, xz, and yz terms influences only one of the midpoints, so they can be matched as well.

Now we show that this particular f can be realized by a choice of triangles. To do this, note that by continuity there must exist x_0, y_0, and z_0 with $f(x_0, y_0, z_0) = 0$, since $f(1, 0, 0)$ and $f(\frac{1}{2}, \frac{1}{2}, 0)$ are different signs, and introduce the new coordinates $u = x - x_0$ and $v = y - y_0$; then f can be written as $au^2 + buv + cv^2 + du + ev$. Now, one can let $P_1(u, v) = (0, 0)$, $P_2(u, v) = (u, v)$, and $P_3(u, v) = (-cv - e, au + bv + d)$. This can be shown to reproduce the desired f.

Finally, to address the condition that the original triangles must be nondegenerate, we can perturb each of the P_i by a constant, which doesn't affect f as areas are translation-invariant. This concludes the proof.

10. Suppose ω is a circle centered at O with radius 8. Let AC and BD be perpendicular chords of ω. Let P be a point inside quadrilateral $ABCD$ such that the circumcircles of triangles ABP and CDP are tangent, and the circumcircles of triangles ADP and BCP are tangent. If $AC = 2\sqrt{61}$ and $BD = 6\sqrt{7}$, then OP can be expressed as $\sqrt{a} - \sqrt{b}$ for positive integers a and b. Compute $100a + b$.

Proposed by: Daniel Xianzhe Hong

Answer: $\boxed{103360}$

Solution: Let $X = AC \cap BD$, $Q = AB \cap CD$ and $R = BC \cap AD$. Since $QA \cdot QB = QC \cdot QD$, Q is on the radical axis of (ABP) and (CDP), so Q lies on the common tangent at P. Thus, $QP^2 = QA \cdot QB$. Similarly, $RA \cdot RC = RP^2$. Let M be the Miquel point of quadrilateral $ABCD$: in particular, $M = OX \cap QR$ is the foot from O to QR. By properties of the Miquel point, $ABMR$ and $ACMQ$ are cyclic. Thus,

$$\begin{aligned} QP^2 &= QA \cdot QB \\ RP^2 &= RA \cdot RC \\ QP^2 + RP^2 &= QM \cdot QR + RM \cdot RQ = (QR + RM)QR = QR^2. \end{aligned}$$

As a result, $\angle QPR = 90°$.

Now, let P' the inverse of P with respect to ω. Note that by properties of inversion, (ABP') and (CDP') are tangent, and (ACP') and (BDP') are also tangent.

But now,

$$\begin{aligned} QP^2 &= QP'^2 = QA \cdot QB \\ RP^2 &= RP'^2 = RA \cdot RC \\ QP^2 + RP^2 &= QP'^2 + RP'^2 = QR^2. \end{aligned}$$

Thus, $PQP'R$ is a cyclic kite, so P and P' are reflections of each other across QR. In particular, since O, P, P' are collinear, then M lies on line OPP'.

We can now compute OP by using the fact that $OP + \frac{r^2}{OP} = 2OM = \frac{2r^2}{OX}$, where $r = 8$. Since OX can be computed to equal 2 quite easily, then $OP + \frac{64}{OP} = 64$, or $OP^2 - 64OP + 64 = 0$. Solving this yields $OP = 32 \pm 8\sqrt{15}$, and because P is inside the circle, $OP = 32 - 8\sqrt{15} = \sqrt{1024} - \sqrt{960}$.