HMMT February 2022

February 19, 2022

Team Round

1. **[20]** Let $(a_1, a_2, \dots, a_8)$ be a permutation of $(1, 2, \dots, 8)$. Find, with proof, the maximum possible number of elements of the set

$$\{a_1, a_1 + a_2, \dots, a_1 + a_2 + \cdots + a_8\}$$

that can be perfect squares.

2. **[25]** Find, with proof, the maximum positive integer k for which it is possible to color $6k$ cells of 6×6 grid such that, for any choice of three distinct rows R_1, R_2, R_3 and three distinct columns C_1, C_2, C_3, there exists an uncolored cell c and integers $1 \le i, j \le 3$ so that c lies in R_i and C_j.

3. **[25]** Let triangle ABC be an acute triangle with circumcircle Γ. Let X and Y be the midpoints of minor arcs $\widehat{AB}$ and $\widehat{AC}$ of Γ, respectively. If line XY is tangent to the incircle of triangle ABC and the radius of Γ is R, find, with proof, the value of XY in terms of R.

4. **[30]** Suppose $n \ge 3$ is a positive integer. Let $a_1 < a_2 < \cdots < a_n$ be an increasing sequence of positive real numbers, and let $a_{n+1} = a_1$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{a_{k+1}} > \sum_{k=1}^{n} \frac{a_{k+1}}{a_k}.$$

5. **[40]** Let ABC be a triangle with centroid G, and let E and F be points on side BC such that $BE = EF = FC$. Points X and Y lie on lines AB and AC, respectively, so that X, Y, and G are not collinear. If the line through E parallel to XG and the line through F parallel to YG intersect at $P \neq G$, prove that GP passes through the midpoint of XY.

6. **[45]** Let $P(x) = x^4 + ax^3 + bx^2 + x$ be a polynomial with four distinct roots that lie on a circle in the complex plane. Prove that $ab \neq 9$.

7. **[50]** Find, with proof, all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that

$$f(x)^2 - f(y)f(z) = x(x + y + z)(f(x) + f(y) + f(z))$$

for all real x, y, z such that $xyz = 1$.

8. **[50]** Let $P_1 P_2 \cdots P_n$ be a regular n-gon in the plane and $a_1, \dots, a_n$ be nonnegative integers. It is possible to draw m circles so that for each $1 \le i \le n$, there are exactly a_i circles that contain P_i on their interior. Find, with proof, the minimum possible value of m in terms of the a_i.

9. **[55]** Let Γ_1 and Γ_2 be two circles externally tangent to each other at N that are both internally tangent to Γ at points U and V, respectively. A common external tangent of Γ_1 and Γ_2 is tangent to Γ_1 and Γ_2 at P and Q, respectively, and intersects Γ at points X and Y. Let M be the midpoint of the arc $\widehat{XY}$ that does not contain U and V. Let Z be on Γ such $MZ \perp NZ$, and suppose the circumcircles of QVZ and PUZ intersect at $T \neq Z$. Find, with proof, the value of $TU + TV$, in terms of R, r_1, and r_2, the radii of Γ, Γ_1, and Γ_2, respectively.

10. **[60]** On a board the following six vectors are written:

$$(1, 0, 0), \quad (-1, 0, 0), \quad (0, 1, 0), \quad (0, -1, 0), \quad (0, 0, 1), \quad (0, 0, -1).$$

Given two vectors v and w on the board, a move consists of erasing v and w and replacing them with $\frac{1}{\sqrt{2}}(v + w)$ and $\frac{1}{\sqrt{2}}(v - w)$. After some number of moves, the sum of the six vectors on the board is u. Find, with proof, the maximum possible length of u.