HMIC 2022

March 31–April 6, 2022

1. **[6]** Is
$$\prod_{k=0}^{\infty}\left(1-\frac{1}{2022^{k!}}\right)$$
rational?

2. **[6]** Does there exist a regular pentagon whose vertices lie on the edges of a cube?

3. **[8]** For a nonnegative integer n, let $s(n)$ be the sum of the digits of the binary representation of n. Prove that
$$\sum_{n=0}^{2^{2022}-1}\frac{(-1)^{s(n)}}{2022+n}>0.$$

4. **[10]** Call a simple graph G *quasicolorable* if we can color each edge blue, red, green, or white such that

   - for each vertex v of degree 3 in G, the three edges incident to v are either (1) red, green, and blue, or (2) all white,
   - not all edges are white.

   A simple connected graph G has a vertices of degree 4, b vertices of degree 3, and no other vertices, where a and b are positive integers. Find the smallest real number c so that the following statement is true: "If $a/b>c$, then G must be quasicolorable."

5. **[12]** Let p be a prime and let $\mathbb{F}_p$ be the set of integers modulo p. Call a function $f\colon \mathbb{F}_p^2\to\mathbb{F}_p$ *quasiperiodic* if there exist $a,b\in\mathbb{F}_p$, not both zero, so that $f(x+a,y+b)=f(x,y)$ for all $x,y\in\mathbb{F}_p$. Find the number of functions $\mathbb{F}_p^2\to\mathbb{F}_p$ that can be written as the sum of some number of quasiperiodic functions.