HMIC 2022

March 31–April 6, 2022

1. **[6]** Is

$$\prod_{k=0}^{\infty}\left(1-\frac{1}{2022^{k!}}\right)$$

rational?

Proposed by: Michael Ren

Answer: $\boxed{\text{No}}$

Solution: It suffices to prove that the product $A=\prod_{k=1}^{\infty}\left(1-\frac{1}{2022^{k!}}\right)$ is irrational. Suppose for the sake of contradiction that A is rational. Note that for each non-negative integer n, there exists at most one subset S of $\{1!,2!,3!,\dots\}$ such that the sum of entries of S equals n. Thus, we have

$$A=\prod_{k=1}^{\infty}\left(1-\frac{1}{2022^{k!}}\right)=\sum_{n=0}^{\infty}\frac{\epsilon_n}{2022^n},$$

where $\epsilon_n\in\{-1,0,1\}$ for each non-negative integer n. Adding $\frac{2022}{2021}=\sum_{n=0}^{\infty}\frac{1}{2022^n}$ to both sides gives us

$$A+\frac{2022}{2021}=\sum_{n=0}^{\infty}\frac{\epsilon'_n}{2022^n},$$

where $\epsilon'_n\in\{0,1,2\}$. Since we assumed A is rational, we have that the sequence $\epsilon'_0,\epsilon'_1,\epsilon'_2,\dots$ must be eventually periodic with period length P, which means that the sequence $\epsilon_0,\epsilon_1,\epsilon_2,\dots$ must also be eventually periodic with period length P. Note that this sequence has infinitely nonzero terms. That means that there exists some positive integer M such that if $n\geq M$, then at least one of $\epsilon_{n+1},\epsilon_{n+2},\dots,\epsilon_{n+P}$ is nonzero. Now, select a positive integer

$$N=T!+(T-1)!+(T-2)!+\cdots+1!,$$

where $T=\max(2P,M)$. Note that $T+1,T+2,\dots,T+P$ all cannot be expressed as the sum of elements of a subset of $\{1!,2!,3!,\dots\}$, which means that $\epsilon_{T+1}=\epsilon_{T+2}=\cdots=\epsilon_{T+P}=0$. Therefore, we have a contradiction, so A is not rational.

2. **[6]** Does there exist a regular pentagon whose vertices lie on edges of a cube?

Proposed by: Akash Das

Answer: $\boxed{\text{No}}$

Solution: If two of the sides of the pentagon lie on the same face of the cube, then we have that we have that all sides of the pentagon lie on this face, which would mean that one could choose five points on the boundary of the unit square that forms a pentagon. This is impossible by the following argument. By pigeonhole, there must be a pair of points that are on the same side of the square (WLOG, say it's the bottom side of the square). If this is the only such pair of points, then the other three points must be on the other three sides in such a way that the pentagon is symmetric with respect to the vertical line passing through the top vertex. But this implies that the height of the pentagon (which is also the height of the square) is equal to the length of one of the diagonals of the pentagon (because it is the width of the square), which is false. The other case is that there is another pair of points that are on the same side of the square. This is impossible because it would imply that two of the sides of the pentagon are parallel or perpendicular (which is false because all angles are multiples of $2\pi/5$). Thus, no two sides of the pentagon lie on the same face.

Note that if each side of the pentagon lies on a face, then because there are four pairs of parallel faces, we know that two sides of the pentagon must be on parallel faces, which mean they are parallel, which can't happen.

Thus, some side of the pentagon must not lie on a face of the cube. The endpoints of this side must have a difference of 1 in one of their coordinates, so the side length of the pentagon must be greater than 1 (and its diagonal has length greater than $\frac{1+\sqrt{5}}{2}$. This tells us that no three vertices of the pentagon can lie on the same face of the cube because a pair of these vertices must be a diagonal of the pentagon, and the greatest distance between two points on a unit square is $\sqrt{2} < \frac{1+\sqrt{5}}{2}$. Further, if two vertices of the pentagon lie on the same face of the cube, the line segment connecting them must be a side of the pentagon.

Each vertex of the pentagon lies on at least two faces, so by pigeonhole (the vertices get counted a total of 10 times across 6 faces), there must be at least four faces with two vertices on them. These correspond to four edges of the pentagon on distinct faces of the cube. But this must include a pair of faces that are parallel to each other, so two of the edges of the pentagon are parallel. This is impossible, so we are done.

3. **[8]** For a nonnegative integer n, let $s(n)$ be the sum of digits of the binary representation of n. Prove that

$$\sum_{n=0}^{2^{2022}-1} \frac{(-1)^{s(n)}}{2022+n} > 0.$$

Proposed by: Akash Das

Solution 1: Define

$$f_k(x) = \sum_{n=0}^{2^k-1} \frac{(-1)^{s(n)}}{x+n}.$$

We want to show that $f_{2022}(2022) > 0$. We will in fact show something stronger.

I claim that for all $x > 0$, for all $k \geq 0$, we have $f_k^{(i)}(x) > 0$ for even i and $f_k^{(i)}(x) < 0$ for odd i, where $f^{(i)}$ denotes the ith derivative of f. We will prove this claim with induction on k. The base case of $k = 0$ is easy to see because $f_0(x) = \frac{1}{x}$, so $f_0^{(2j)}(x) = \frac{(2j)!}{x^{2j+1}} > 0$ and $f_0^{(2j-1)}(x) = -\frac{(2j-1)!}{x^{2j}} < 0$ for all $x > 0$. Now, assume the claim is true for $k = N$. Then, note that

$$f_{N+1}(x) = \sum_{n=0}^{2^{N+1}-1} \frac{(-1)^{s(n)}}{x+n} = \sum_{n=0}^{2^N-1} \frac{(-1)^{s(n)}}{x+n} + \sum_{n=0}^{2^N-1} \frac{(-1)^{s(n+2^N)}}{x+n+2^N} =$$

$$\sum_{n=0}^{2^N-1} \frac{(-1)^{s(n)}}{x+n} - \sum_{n=0}^{2^N-1} \frac{(-1)^{s(n)}}{x+n+2^N} = f_N(x) - f_N(x+2^N).$$

Thus,

$$f_{N+1}^{(2j)}(x) = f_N^{(2j)}(x) - f_N^{(2j)}(x+2^N) > 0,$$

since $\left(f_N^{(2j)}(x)\right)' = f_N^{(2j+1)}(x) < 0$. Similarly, we can show that $f_{N+1}^{2j+1}(x) < 0$, which completes the induction, so we are done.

Solution 2: Define the function

$$f(t) = t^{2021}(1-t)(1-t^2)(1-t^4)(1-t^8)\cdots(1-t^{2^{2021}}) = \sum_{n=0}^{2^{2022}-1} (-1)^{s(n)} t^{2021+n}.$$

Note that we have $f(t) > 0$ for all $t \in (0,1)$, so we have

$$0 < \int_0^1 f(t)dt = \sum_{n=0}^{2^{2022}-1} (-1)^{s(n)} \frac{1}{2022+n},$$

so we are done.

4. **[10]** Call a simple graph G *quasi-colorable* if we can color each edge blue, red, green, and white such that

   - for each vertex v of degree 3 in G, the three edges containing v as an endpoint are either colored blue, red, and green, or all three edges are white,
   - not all edges are white.

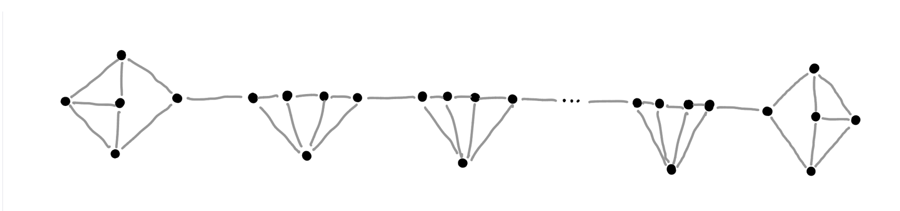
   A connected graph G has a vertices of degree 4, b vertices of degree 3, and no other vertices, where a and b are positive integers. Find the smallest real number c so that the following statement is true: "If $a/b > c$, then G is quasi-colorable."

   Proposed by: Akash Das

   Answer: $\boxed{1/4}$

   Solution 1: Consider a graph G such that $\frac{a}{b} > \frac{1}{4}$. Note that the number of edges is $\frac{4a+3b}{2}$. Additionally, if any two vertices of degree 4 are adjacent, we can simply color that edge red and every other edge in G white to get a valid quasi-coloring. Thus, suppose no two vertices of degree 4 are adjacent. Then, consider the subgraph G' resulting from removing all vertices of degree 4. Note that G' has $\frac{4a+3b}{2} - 4a = \frac{3b-4a}{2} < \frac{3b-b}{2} = b$ edges. Thus, G' has fewer edges than vertices, which means that one of it's connected components must be a tree T. In this tree T, we can just select some arbitrary vertex v, and perform a breadth-first search on T, greedily coloring the edges as we go along. What this means is that we partition the vertices of T into disjoint subsets $S_0 = \{v\}, S_1, S_2, \dots,$ where S_j contains all vertices of T whose distance to v is exactly j, and then we color edges between S_0 and S_1, and then greedily color the edges between S_1 and S_2, and so on. Once we color all the edges in T, we can color the edges between vertices in T and the vertices of degree 4 appropriately, and color the remaining edges white to get a valid quasi-coloring.

   To show that $c = \frac{1}{4}$ is indeed the smallest possible solution, consider the graph shown below. This graph is not quasi-colorable (one can see that coloring the tails of this graph is impossible).



   If there are N degree vertices of degree 4, there are $4N + 10$ vertices of degree 3. Since $\frac{N}{4N+10} \to \frac{1}{4}$ as $N \to \infty$, we have that for each value $c < \frac{1}{4}$, one can find a graph with $a = N_0$ vertices of degree 4 and $b = 4N_0 + 10$ vertices of degree 3 such that $\frac{a}{b} = \frac{N_0}{4N_0+10} > c$.

   Solution 2: Consider a graph G such that $\frac{a}{b} > \frac{1}{4} \iff b < 4a$. Note that the number of edges is $\frac{4a+3b}{2}$. For each edge e in the graph, create a variable x_e in $\mathbb{F}_7$. Then, for each vertex v of degree 3, if the edges incident with v are e_1, e_2, and e_3, then consider polynomial

$$P_v = x_{e_1}^2 + x_{e_2}^2 + x_{e_3}^2$$

   in $\mathbb{F}_7$. There are b such polynomials, each with total degree 2. Thus, the since $b < 4a$, we know that the sum of the total degrees over all polynomials, $2b$, is less than the total number of variables, $\frac{4a+3b}{2}$. By Chevalley's Theorem, it follows that there exists a nontrivial solution to the system $P_v \equiv 0 \pmod 7$

for all vertices v of degree 3. Consider one such solution. For each edge e, if $x_e \equiv 0 \pmod 7$, then color edge e white. If $x_e^2 \equiv 1 \pmod 7$, color edge e blue. If $x_e^2 \equiv 2 \pmod 7$, color edge e red. If $x_e^2 \equiv 4 \pmod 7$, color edge e green. It is not hard to see that this is a valid quasi-coloring.

To show that $c = \frac{1}{4}$ is indeed the smallest possible solution, we can use the graph shown in the previous solution.

5. **[12]** Let p be a prime and let $\mathbb{F}_p$ be the set of integers modulo p. Call a function $f\colon \mathbb{F}_p^2 \to \mathbb{F}_p$ quasiperiodic if there exist $a, b \in \mathbb{F}_p$, not both zero, so that $f(x+a, y+b) = f(x,y)$ for all $x, y \in \mathbb{F}_p$. Find, with proof, the number of functions $\mathbb{F}_p^2 \to \mathbb{F}_p$ that can be written as the sum of some number of quasiperiodic functions.

Note: The original version of this problem neglected to mention that p was a prime. We regret the error.

Proposed by: Daniel Zhu

Answer: $\boxed{p^{\frac{p(p+1)}{2}}}$

Solution 1: Every function $\mathbb{F}_p^2 \to \mathbb{F}_p$ can be written uniquely as a polynomial $\sum_{i=0}^{p-1}\sum_{j=0}^{p-1} a_{ij}x^i y^j$. We claim that a function can be written as a sum of quasiperiodic functions if and only if $a_{ij} = 0$ for all $i + j \geq p$. The only if direction follows directly from the fact that quasiperiodic functions are all of the form $g(by - ax)$ for some function g with can be written as a polynomial of degree at most $p-1$.

The if direction is more involved. Pick some $d \leq p-1$, and we will argue by induction on i that $x^i y^{d-i}$ is a sum of quasiperiodic functions. This is clear for $i = 0$. Now, if $i > 0$ consider the following sum of quasiperiodic functions:

$$\begin{aligned}\sum_{j=0}^{d-i}(-1)^j\binom{d-i}{j}(x+jy)^d &= \sum_{k=0}^{d}\sum_{j=0}^{d-i}(-1)^j\binom{d-i}{j}\binom{d}{k}x^k j^{d-k} y^{d-k}\\ &= \sum_{k=0}^{d}\binom{d}{k}x^k y^{d-k}\sum_{j=0}^{d-i}(-1)^j\binom{d-i}{j}j^{d-k}.\end{aligned}$$

By the theory of finite differences, we know that $\sum_{j=0}^{d-i}(-1)^j\binom{d-i}{j}j^{d-k}$ is zero if $d-i > d-k \iff k > i$, and nonzero if $d-i = d-k \iff i = k$ (specifically it is $(-1)^{d-i}(d-i)!$). Therefore, this is

$$\sum_{k=0}^{i} a_k x^k y^{d-k}$$

for some a_k so that a_i is nonzero. By the inductive hypothesis, $x^k y^{d-k}$ is a sum of quasiperiodic functions for all $k < i$, so we are done.

Solution 2 (loosely based on student submission)**:** This solution will make heavy use of linear algebra. First, let $\mathcal{L}$ be the set of affine lines in $\mathbb{F}_p^2$. If $\ell \in \mathcal{L}$ and $f\colon \mathbb{F}_p^2 \to \mathbb{F}_p$ is a function, let $S_\ell(f) = \sum_{(x,y)\in\ell} f(x,y)$ and let $\mathbf{1}_\ell\colon \mathbb{F}_p^2 \to \mathbb{F}_p$ be the indicator function of ℓ. Let V be the vector space spanned by all the quasiperiodic functions, which can alternatively defined by the space spanned by all the $\mathbf{1}_\ell$. Furthermore, let W be the vector space of functions f so that $S_\ell(f) = 0$ for all lines ℓ.

This next claim is the heart of the proof:

Claim. $W \subseteq V$.

Proof. Surprisingly, this is easier to prove when lifted to $\mathbb{Z}$. Specifically, for a line $\ell \subseteq \mathbb{F}_p^2$, let $\tilde{\mathbf{1}}_\ell\colon \mathbb{F}_p^2 \to \mathbb{Z}$ be the indicator function of ℓ and let $f\colon \mathbb{F}_p^2 \to \mathbb{Z}$ be a function so that $S_\ell(f) \equiv 0 \pmod p$ for all ℓ. We will show that f can be written as a linear combination of functions of the form $\tilde{\mathbf{1}}_\ell$ and a remainder term g so that $g(x,y) \equiv 0 \pmod p$. (It is in fact true that g can be made to be identically zero, but this is irrelevant for our purposes.)

Suppose that a counterexample f exists. Then, by possibly adjusting the values of f by multiples by p (which can be absorbed into the remainder g term), we may assume that $\sum_{x,y\in\mathbb{F}_p} f(x,y) = 0$. Out of all such counterexamples, pick an f that minimizes the sum $\sum_{\ell\in\mathcal{L}}|S_\ell(f)|$.

We now split the analysis into two cases. If $S_\ell(f) = 0$ for all lines ℓ, then for every point $(x,y) \in \mathbb{F}_p^2$ we have

$$pf(x,y) = \sum_{(x,y)\in\ell} S_\ell(f) - \sum_{x',y'\in\mathbb{F}_p^2} f(x',y') = 0,$$

so f is identically zero. This clearly cannot be a counterexample.

In the other case, we must have a line ℓ with $S_\ell(f) > 0$. Since the sum of all values of f is zero, there is a line ℓ' parallel to ℓ with $S_{\ell'}(f) < 0$. Then it is not hard to show that $f' = f - \tilde{\mathbf{1}}_\ell + \tilde{\mathbf{1}}_{\ell'}$ is another counterexample with $\sum_{\ell\in\mathcal{L}}|S_\ell(f')| < \sum_{\ell\in\mathcal{L}}|S_\ell(f)|$, which contradicts the minimality of f. This finishes the proof of the claim. ■

Because of the claim, we can now describe W as the kernel of the map $\Phi\colon V \to \mathbb{F}_p^{\mathcal{L}}$ sending f to $(S_\ell(f))_{\ell\in\mathcal{L}}$. Note that $S_\ell(\mathbf{1}_{\ell'})$ is 0 if ℓ and ℓ' are parallel and 1 otherwise, so the image of Φ is generated by the $p+1$ vectors $v_i = \Phi(\mathbf{1}_{\ell_i})$, where $\ell_1, \ell_2, \dots, \ell_{p+1}$ range over the parallel classes of lines in $\mathbb{F}_p^2$. One can check that $v_1 + v_2 + \cdots + v_{p+1} = 0$, but this is in fact the only linear relation. To see this, note that if $\sum_i c_i v_i = 0$, then for all j we have $0 = \sum_i c_i S_{\ell_j}(\mathbf{1}_{\ell_i}) = -c_j + \sum_i c_i$. It follows that all the c_j are the same. Therefore, the image of Φ has dimension p.

Since the image of Φ has dimension p, we conclude that $\dim V - \dim W = p$. Also, by definition, W is space of functions that are orthogonal to V under the bilinear pairing $(f,g) \mapsto \sum_{x,y\in\mathbb{F}_p} f(x,y)g(x,y)$, so $\dim V + \dim W = p^2$. Solving these equations yields that $\dim V = \frac{p(p+1)}{2}$, so the size of V is $p^{\frac{p(p+1)}{2}}$.