HMMT November

November 12, 2022

General Round

1. Emily's broken clock runs backwards at five times the speed of a regular clock. Right now, it is displaying the wrong time. How many times will it display the correct time in the next 24 hours? It is an analog clock (i.e. a clock with hands), so it only displays the numerical time, not AM or PM. Emily's clock also does not tick, but rather updates continuously.

2. How many ways are there to arrange the numbers $1, 2, 3, 4, 5, 6$ on the vertices of a regular hexagon such that exactly 3 of the numbers are larger than both of their neighbors? Rotations and reflections are considered the same.

3. Let $ABCD$ be a rectangle with $AB = 8$ and $AD = 20$. Two circles of radius 5 are drawn with centers in the interior of the rectangle - one tangent to AB and AD, and the other passing through both C and D. What is the area inside the rectangle and outside of both circles?

4. Let $x < 0.1$ be a positive real number. Let the *foury series* be $4 + 4x + 4x^2 + 4x^3 + \dots$, and let the *fourier series* be $4 + 44x + 444x^2 + 4444x^3 + \dots$. Suppose that the sum of the fourier series is four times the sum of the foury series. Compute x.

5. An apartment building consists of 20 rooms numbered $1, 2, \dots, 20$ arranged clockwise in a circle. To move from one room to another, one can either walk to the next room clockwise (i.e. from room i to room $(i + 1) \pmod{20}$) or walk across the center to the opposite room (i.e. from room i to room $(i + 10) \pmod{20}$). Find the number of ways to move from room 10 to room 20 without visiting the same room twice.

6. In a plane, equilateral triangle ABC, square $BCDE$, and regular dodecagon $DEFGHIJKLMNO$ each have side length 1 and do not overlap. Find the area of the circumcircle of $\triangle AFN$.

7. In circle ω, two perpendicular chords intersect at a point P. The two chords have midpoints M_1 and M_2 respectively, such that $PM_1 = 15$ and $PM_2 = 20$. Line M_1M_2 intersects ω at points A and B, with M_1 between A and M_2. Compute the largest possible value of $BM_2 - AM_1$.

8. Compute the number of sets S such that every element of S is a nonnegative integer less than 16, and if $x \in S$ then $(2x \bmod 16) \in S$.

9. Call a positive integer n *quixotic* if the value of
$$\operatorname{lcm}(1, 2, 3, \dots, n) \cdot \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$$
is divisible by 45. Compute the tenth smallest quixotic integer.

10. Compute the number of distinct pairs of the form
$$(\text{first three digits of } x,\ \text{first three digits of } x^4)$$
over all integers $x > 10^{10}$.

    For example, one such pair is $(100, 100)$ when $x = 10^{10^{10}}$.