HMMT November 2022, November 12, 2022 — GUTS ROUND

Organization _____________________________ Team _____________________________ Team ID# ___________

1. **[5]** Compute $\sqrt{2022^2 - 12^6}$.

2. **[5]** The English alphabet, which has 26 letters, is randomly permuted. Let p_1 be the probability that `AB`, `CD`, and `EF` all appear as contiguous substrings. Let p_2 be the probability that `ABC` and `DEF` both appear as contiguous substrings. Compute $\frac{p_1}{p_2}$.

3. **[5]** A polygon $\mathcal{P}$ is drawn on the 2D coordinate plane. Each side of $\mathcal{P}$ is either parallel to the x axis or the y axis (the vertices of $\mathcal{P}$ do not have to be lattice points). Given that the interior of $\mathcal{P}$ includes the interior of the circle $x^2 + y^2 = 2022$, find the minimum possible perimeter of $\mathcal{P}$.

HMMT November 2022, November 12, 2022 — GUTS ROUND

Organization _____________________________ Team _____________________________ Team ID# ___________

4. **[6]** Let $ABCD$ be a square of side length 2. Let points X, Y, and Z be constructed inside $ABCD$ such that ABX, BCY, and CDZ are equilateral triangles. Let point W be outside $ABCD$ such that triangle DAW is equilateral. Let the area of quadrilateral $WXYZ$ be $a + \sqrt{b}$, where a and b are integers. Find $a + b$.

5. **[6]** Suppose x and y are positive real numbers such that

$$x + \frac{1}{y} = y + \frac{2}{x} = 3.$$

Compute the maximum possible value of xy.

6. **[6]** Let $ABCDEF$ be a regular hexagon and let point O be the center of the hexagon. How many ways can you color these seven points either red or blue such that there doesn't exist any equilateral triangle with vertices of all the same color?

HMMT November 2022, November 12, 2022 — GUTS ROUND

Organization _____________________________ Team _____________________________ Team ID# ___________

7. **[7]** All positive integers whose binary representations (excluding leading zeroes) have at least as many 1's as 0's are put in increasing order. Compute the number of digits in the binary representation of the 200th number.

8. **[7]** Kimothy starts in the bottom-left square of a 4 by 4 chessboard. In one step, he can move up, down, left, or right to an adjacent square. Kimothy takes 16 steps and ends up where he started, visiting each square exactly once (except for his starting/ending square). How many paths could he have taken?

9. **[7]** Let $ABCD$ be a trapezoid such that $AB \parallel CD$, $\angle BAC = 25°$, $\angle ABC = 125°$, and $AB + AD = CD$. Compute $\angle ADC$.

Organization ________________ Team ________________ Team ID# ________

10. **[8]** A real number x is chosen uniformly at random from the interval $[0, 1000]$. Find the probability that

$$\left\lfloor \frac{\left\lfloor \frac{x}{2.5} \right\rfloor}{2.5} \right\rfloor = \left\lfloor \frac{x}{6.25} \right\rfloor.$$

11. **[8]** Isosceles trapezoid $ABCD$ with bases AB and CD has a point P on AB with $AP = 11$, $BP = 27$, $CD = 34$, and $\angle CPD = 90^\circ$. Compute the height of isosceles trapezoid $ABCD$.

12. **[8]** Candice starts driving home from work at 5:00 PM. Starting at exactly 5:01 PM, and every minute after that, Candice encounters a new speed limit sign and slows down by 1 mph. Candice's speed, in miles per hour, is always a positive integer. Candice drives for $2/3$ of a mile in total. She drives for a whole number of minutes, and arrives at her house driving slower than when she left. What time is it when she gets home?

HMMT November 2022, November 12, 2022 — GUTS ROUND

Organization ________________ Team ________________ Team ID# ________

13. **[9]** Consider the paths from $(0, 0)$ to $(6, 3)$ that only take steps of unit length up and right. Compute the sum of the areas bounded by the path, the x-axis, and the line $x = 6$ over all such paths.

    (In particular, the path from $(0, 0)$ to $(6, 0)$ to $(6, 3)$ corresponds to an area of 0.)

14. **[9]** Real numbers x and y satisfy the following equations:

$$x = \log_{10}(10^{y-1} + 1) - 1$$
$$y = \log_{10}(10^{x} + 1) - 1.$$

    Compute 10^{x-y}.

15. **[9]** Vijay chooses three distinct integers a, b, c from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. If k is the minimum value taken on by the polynomial $a(x - b)(x - c)$ over all real numbers x, and l is the minimum value taken on by the polynomial $a(x - b)(x + c)$ over all real numbers x, compute the maximum possible value of $k - l$.

HMMT November 2022, November 12, 2022 — GUTS ROUND

Organization ________________ Team ________________ Team ID# ________

16. **[10]** Given an angle θ, consider the polynomial

$$P(x) = \sin(\theta)x^2 + (\cos(\theta) + \tan(\theta))x + 1.$$

    Given that P only has one real root, find all possible values of $\sin(\theta)$.

17. **[10]** How many ways are there to color every integer either red or blue such that n and $n + 7$ are the same color for all integers n, and there does not exist an integer k such that k, $k + 1$, and $2k$ are all the same color?

18. **[10]** A regular tetrahedron has a square shadow of area 16 when projected onto a flat surface (light is shone perpendicular onto the plane). Compute the sidelength of the regular tetrahedron.

    (For example, the shadow of a sphere with radius 1 onto a flat surface is a disk of radius 1.)

Organization _______________________ Team _______________________ Team ID# _________

19. **[11]** Define the *annoyingness* of a permutation of the first n integers to be the minimum number of copies of the permutation that are needed to be placed next to each other so that the subsequence $1, 2, \ldots, n$ appears. For instance, the annoyingness of $3, 2, 1$ is 3, and the annoyingness of $1, 3, 4, 2$ is 2.

    A random permutation of $1, 2, \ldots, 2022$ is selected. Compute the expected value of the annoyingness of this permutation.

20. **[11]** Let $\triangle ABC$ be an isosceles right triangle with $AB = AC = 10$. Let M be the midpoint of BC and N the midpoint of BM. Let AN hit the circumcircle of $\triangle ABC$ again at T. Compute the area of $\triangle TBC$.

21. **[11]** Let $P(x)$ be a quadratic polynomial with real coefficients. Suppose that $P(1) = 20$, $P(-1) = 22$, and $P(P(0)) = 400$. Compute the largest possible value of $P(10)$.

HMMT November 2022, November 12, 2022 — GUTS ROUND

Organization _______________________ Team _______________________ Team ID# _________

22. **[12]** Find the number of pairs of integers (a, b) with $1 \leq a < b \leq 57$ such that a^2 has a smaller remainder than b^2 when divided by 57.

23. **[12]** Let ABC be a triangle with $AB = 2021$, $AC = 2022$, and $BC = 2023$. Compute the minimum value of $AP + 2BP + 3CP$ over all points P in the plane.

24. **[12]** A string consisting of letters `A`, `C`, `G`, and `U` is *untranslatable* if and only if it has no `AUG` as a consecutive substring. For example, `ACUGG` is untranslatable.

    Let a_n denote the number of untranslatable strings of length n. It is given that there exists a unique triple of real numbers (x, y, z) such that $a_n = xa_{n-1} + ya_{n-2} + za_{n-3}$ for all integers $n \geq 100$. Compute (x, y, z).

HMMT November 2022, November 12, 2022 — GUTS ROUND

Organization _______________________ Team _______________________ Team ID# _________

25. **[13]** In convex quadrilateral $ABCD$ with $AB = 11$ and $CD = 13$, there is a point P for which $\triangle ADP$ and $\triangle BCP$ are congruent equilateral triangles. Compute the side length of these triangles.

26. **[13]** A number is chosen uniformly at random from the set of all positive integers with at least two digits, none of which are repeated. Find the probability that the number is even.

27. **[13]** How many ways are there to cut a 1 by 1 square into 8 congruent polygonal pieces such that all of the interior angles for each piece are either 45 or 90 degrees? Two ways are considered distinct if they require cutting the square in different locations. In particular, rotations and reflections are considered distinct.

Organization ____________________ Team ____________________ Team ID# __________

28. **[15]** Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Pick points Q and R on AC and AB such that $\angle CBQ = \angle BCR = 90^\circ$. There exist two points $P_1 \neq P_2$ in the plane of ABC such that $\triangle P_1QR$, $\triangle P_2QR$, and $\triangle ABC$ are similar (with vertices in order). Compute the sum of the distances from P_1 to BC and P_2 to BC.

29. **[15]** Consider the set S of all complex numbers z with nonnegative real and imaginary part such that

$$|z^2 + 2| \leq |z|.$$

Across all $z \in S$, compute the minimum possible value of $\tan\theta$, where θ is the angle formed between z and the real axis.

30. **[15]** Let ABC be a triangle with $AB = 8$, $AC = 12$, and $BC = 5$. Let M be the second intersection of the internal angle bisector of $\angle BAC$ with the circumcircle of ABC. Let ω be the circle centered at M tangent to AB and AC. The tangents to ω from B and C, other than AB and AC respectively, intersect at a point D. Compute AD.

HMMT November 2022, November 12, 2022 — GUTS ROUND

Organization ____________________ Team ____________________ Team ID# __________

31. **[17]** Given positive integers $a_1, a_2, \ldots, a_{2023}$ such that

$$a_k = \sum_{i=1}^{2023} |a_k - a_i|$$

for all $1 \leq k \leq 2023$, find the minimum possible value of $a_1 + a_2 + \cdots + a_{2023}$.

32. **[17]** Suppose point P is inside triangle ABC. Let AP, BP, and CP intersect sides BC, CA, and AB at points D, E, and F, respectively. Suppose $\angle APB = \angle BPC = \angle CPA$, $PD = \frac{1}{4}$, $PE = \frac{1}{5}$, and $PF = \frac{1}{7}$. Compute $AP + BP + CP$.

33. **[17]** A group of 101 Dalmathians participate in an election, where they each vote independently on either candidate A or B with equal probability. If X Dalmathians voted for the winning candidate, the expected value of X^2 can be expressed as $\frac{a}{b}$ for positive integers a, b with $\gcd(a, b) = 1$. Find the unique positive integer $k \leq 103$ such that $103 \mid a - bk$.

HMMT November 2022, November 12, 2022 — GUTS ROUND

Organization ____________________ Team ____________________ Team ID# __________

34. **[20]** A random binary string of length 1000 is chosen. Let L be the expected length of its longest (contiguous) palindromic substring. Estimate L.

    An estimate of E will receive $\lfloor 20\min(\frac{E}{L}, \frac{L}{E})^{10} \rfloor$ points.

35. **[20]** For each $i \in \{1, \ldots, 10\}$, a_i is chosen independently and uniformly at random from $[0, i^2]$. Let P be the probability that $a_1 < a_2 < \cdots < a_{10}$. Estimate P.

    An estimate of E will earn $\lfloor 20\min(\frac{E}{P}, \frac{P}{E}) \rfloor$ points.

36. **[20]** Consider all questions on this year's contest that ask for a single real-valued answer (excluding this one). Let M be the median of these answers. Estimate M.

    An estimate of E will earn $\lfloor 20\min(\frac{E}{M}, \frac{M}{E})^4 \rfloor$ points.