HMMT November 2022

November 12, 2022

Guts Round

1. **[5]** Compute $\sqrt{2022^2 - 12^6}$.

   Proposed by: Luke Robitaille

   Answer: $\boxed{1050}$

   Solution: Compute

$$\begin{aligned} 2022^2 - 12^6 &= (2022 - 12^3)(2022 + 12^3) \\ &= 294 \cdot 3750 \\ &= (2 \cdot 3 \cdot 7^2)(2 \cdot 3 \cdot 5^4), \end{aligned}$$

   so the answer is $2 \cdot 3 \cdot 5^2 \cdot 7 = 1050$.

2. **[5]** The English alphabet, which has 26 letters, is randomly permuted. Let p_1 be the probability that `AB`, `CD`, and `EF` all appear as contiguous substrings. Let p_2 be the probability that `ABC` and `DEF` both appear as contiguous substrings. Compute $\frac{p_1}{p_2}$.

   Proposed by: Ankit Bisain, Luke Robitaille

   Answer: $\boxed{23}$

   Solution: There are 23! ways to arrange the alphabet such that `AB`, `CD`, and `EF` all appear as contiguous substrings: treat each of these pairs of letters as a single merged symbol, which leaves 23 symbols to permute. Similarly, there are 22! ways to arrange the alphabet such that `ABC` and `DEF` both appear as contiguous substrings. Thus, $p_1 = 23!/26!$ and $p_2 = 22!/26!$, so the answer is $23!/22! = 23$.

3. **[5]** A polygon $\mathcal{P}$ is drawn on the 2D coordinate plane. Each side of $\mathcal{P}$ is either parallel to the x axis or the y axis (the vertices of $\mathcal{P}$ do not have to be lattice points). Given that the interior of $\mathcal{P}$ includes the interior of the circle $x^2 + y^2 = 2022$, find the minimum possible perimeter of $\mathcal{P}$.

   Proposed by: Carl Schildkraut

   Answer: $\boxed{8\sqrt{2022}}$

   Solution: The minimum possible perimeter is achieved by an axis-aligned square with all four sides tangent to the circle, which has area $8\sqrt{2022}$. To see why this is true, notice that there must be at least $2\sqrt{2022}$ length of total perimeter facing left, $2\sqrt{2022}$ length facing up, $2\sqrt{2022}$ facing right, and $2\sqrt{2022}$ facing down in order for the polygon to be closed and have a shadow of length at least $2\sqrt{2022}$ in both the x and y directions.

4. **[6]** Let $ABCD$ be a square of side length 2. Let points X, Y, and Z be constructed inside $ABCD$ such that ABX, BCY, and CDZ are equilateral triangles. Let point W be outside $ABCD$ such that triangle DAW is equilateral. Let the area of quadrilateral $WXYZ$ be $a + \sqrt{b}$, where a and b are integers. Find $a + b$.

   Proposed by: Kevin Min

   Answer: $\boxed{10}$

   Solution: $WXYZ$ is a kite with diagonals XZ and WY, which have lengths $2\sqrt{3} - 2$ and 2, so the area is $2\sqrt{3} - 2 = \sqrt{12} - 2$.

5. **[6]** Suppose x and y are positive real numbers such that

$$x + \frac{1}{y} = y + \frac{2}{x} = 3.$$

Compute the maximum possible value of xy.

Proposed by: Rishabh Das

Answer: $\boxed{3+\sqrt{7}}$

Solution 1: Rewrite the equations as $xy + 1 = 3y$ and $xy + 2 = 3x$. Let $xy = C$, so $x = \frac{C+2}{3}$ and $y = \frac{C+1}{3}$. Then

$$\left(\frac{C+2}{3}\right)\left(\frac{C+1}{3}\right) = C \implies C^2 - 6C + 2 = 0.$$

The larger of its two roots is $3 + \sqrt{7}$.

Solution 2: Multiply the two equations to get $xy + 3 + \frac{2}{xy} = 9$, so letting $C = xy$ gives $C^2 - 6C + 2 = 0$, which has larger root $C = 3 + \sqrt{7}$.

6. **[6]** Let $ABCDEF$ be a regular hexagon and let point O be the center of the hexagon. How many ways can you color these seven points either red or blue such that there doesn't exist any equilateral triangle with vertices of all the same color?

Proposed by: Evan Erickson

Answer: $\boxed{6}$

Solution: Without loss of generality, let O be blue. Then we can't have any two adjacent blues on the perimeter of $ABCDEF$. However, because of the two larger equilateral triangles ACE and BDF, we need at least two blues to keep us from having an all red equilateral triangle. We can't have three blues on the perimeter without break the rule, so we must have two. With this, they must be diametrically opposite. So, in total, there are $2 \times 3 = 6$ good colorings.

7. **[7]** All positive integers whose binary representations (excluding leading zeroes) have at least as many 1's as 0's are put in increasing order. Compute the number of digits in the binary representation of the 200th number.

Proposed by: Isabella Quan

Answer: $\boxed{9}$

Solution: We do a rough estimation. There are 255 positive integers with at most 8 digits and a majority of them, but not more than 200, satisfy the property. Meanwhile, there are 511 positive integers with at most 9 digits, and a majority of them satisfy this property. Thus, the answer must be greater than 8 and at most 9.

8. **[7]** Kimothy starts in the bottom-left square of a 4 by 4 chessboard. In one step, he can move up, down, left, or right to an adjacent square. Kimothy takes 16 steps and ends up where he started, visiting each square exactly once (except for his starting/ending square). How many paths could he have taken?

Proposed by: Richard Qi

Answer: $\boxed{12}$

Solution: The problem is asking to count the number of cycles on the board that visit each square once. We first count the number of cycle shapes, then multiply by 2 because each shape can be traversed in either direction. Each corner must contain an L-shaped turn, which simplifies the casework. In the end there are only two valid cases: the path must either create a U shape (4 possible orientations) or an H shape (2 possible orientations). Thus, the answer is $2(4+2) = 12$.

9. **[7]** Let $ABCD$ be a trapezoid such that $AB \parallel CD$, $\angle BAC = 25^\circ$, $\angle ABC = 125^\circ$, and $AB + AD = CD$. Compute $\angle ADC$.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{70^\circ}$

   Solution: Construct the parallelogram $ABED$. From the condition $AB + AD = CD$, we get that $EC = AD = EB$. Thus,

$$\angle ADC = \angle BEC = 180^\circ - 2\angle BCE = 180^\circ - 2 \cdot 55^\circ = 70^\circ.$$

10. **[8]** A real number x is chosen uniformly at random from the interval $[0, 1000]$. Find the probability that

$$\left\lfloor \frac{\left\lfloor \frac{x}{2.5} \right\rfloor}{2.5} \right\rfloor = \left\lfloor \frac{x}{6.25} \right\rfloor.$$

   Proposed by: Rishabh Das

   Answer: $\boxed{\frac{9}{10}}$

   Solution: Let $y = \frac{x}{2.5}$, so y is chosen uniformly at random from $[0, 400]$. Then we need

$$\left\lfloor \frac{\lfloor y \rfloor}{2.5} \right\rfloor = \left\lfloor \frac{y}{2.5} \right\rfloor.$$

   Let $y = 5a + b$, where $0 \le b < 5$ and a is an integer. Then

$$\left\lfloor \frac{\lfloor y \rfloor}{2.5} \right\rfloor = \left\lfloor \frac{5a + \lfloor b \rfloor}{2.5} \right\rfloor = 2a + \left\lfloor \frac{\lfloor b \rfloor}{2.5} \right\rfloor$$

   while

$$\left\lfloor \frac{y}{2.5} \right\rfloor = \left\lfloor \frac{5a + b}{2.5} \right\rfloor = 2a + \left\lfloor \frac{b}{2.5} \right\rfloor,$$

   so we need $\left\lfloor \frac{\lfloor b \rfloor}{2.5} \right\rfloor = \left\lfloor \frac{b}{2.5} \right\rfloor$, where b is selected uniformly at random from $[0, 5]$. This can be shown to always hold except for $b \in [2.5, 3)$, so the answer is $1 - \frac{0.5}{5} = \frac{9}{10}$.

11. **[8]** Isosceles trapezoid $ABCD$ with bases AB and CD has a point P on AB with $AP = 11$, $BP = 27$, $CD = 34$, and $\angle CPD = 90^\circ$. Compute the height of isosceles trapezoid $ABCD$.

   Proposed by: Albert Wang

   Answer: $\boxed{15}$

   Solution: Drop projections of A, P, B onto CD to get A', P', B'. Since $A'B' = 38$ and $CD = 34$, we get that $DA' = CB' = 2$. Thus, $P'D = 9$ and $P'C = 25$. Hence, the answer is $PP' = \sqrt{P'D \cdot P'C} = 15$.

12. **[8]** Candice starts driving home from work at 5:00 PM. Starting at exactly 5:01 PM, and every minute after that, Candice encounters a new speed limit sign and slows down by 1 mph. Candice's speed, in miles per hour, is always a positive integer. Candice drives for $2/3$ of a mile in total. She drives for a whole number of minutes, and arrives at her house driving slower than when she left. What time is it when she gets home?

Proposed by: Preston Bushnell

Answer: $\boxed{\text{5:05 (PM)}}$

Solution: Suppose that Candice starts driving at n miles per hour. Then she slows down and drives $(n-1)$ mph, $(n-2)$ mph, and so on, with her last speed being $(m+1)$ mph.

Then the total distance traveled is

$$\begin{aligned}\frac{n}{60}+\frac{n-1}{60}+\cdots+\frac{m+1}{60} &= \frac{1}{60}\left(\frac{n(n+1)}{2}-\frac{m(m+1)}{2}\right)\\ &= \frac{n^2+n-m^2-m}{120}\\ &= \frac{(n+m+1)(n-m)}{120}.\end{aligned}$$

Since the total distance travelled is $2/3$, we have $(n+m+1)(n-m) = 120\cdot 2/3 = 80$. We know m is nonnegative since Candice's speed is always positive, so $n+m+1 > n-m$. Thus, $n+m+1$ and $n-m$ are a factor pair of 80, with $n+m+1$ greater and $n-m$ smaller. Since one is even and one is odd, this means we either have $(n+m+1, n-m) = (80,1)$ or $(16,5)$. The first case is impossible since it gives $n-m=1$, which would imply that Candice drives at n mph the whole way home. Therefore, $(n+m-1, n-m) = (16,5)$. Since $n-m=5$, she gets home at 5:05 pm.

13. **[9]** Consider the paths from $(0,0)$ to $(6,3)$ that only take steps of unit length up and right. Compute the sum of the areas bounded by the path, the x-axis, and the line $x=6$ over all such paths.

(In particular, the path from $(0,0)$ to $(6,0)$ to $(6,3)$ corresponds to an area of 0.)

Proposed by: Andrew Lee

Answer: $\boxed{756}$

Solution: We see that the sum of the areas under the path is equal the sum of the areas above the path. Thus, the sum of the areas under the path is half the area of the rectangle times the number of paths, which is $\frac{18\binom{9}{3}}{2} = 756$.

14. **[9]** Real numbers x and y satisfy the following equations:

$$\begin{aligned}x &= \log_{10}(10^{y-1}+1)-1\\ y &= \log_{10}(10^{x}+1)-1.\end{aligned}$$

Compute 10^{x-y}.

Proposed by: Ankit Bisain

Answer: $\boxed{\frac{101}{110}}$

Solution: Taking 10 to the power of both sides in each equation, these equations become:

$$\begin{aligned}10^x &= \left(10^{y-1}+1\right)\cdot 10^{-1}\\ 10^y &= \left(10^{x}+1\right)\cdot 10^{-1}.\end{aligned}$$

Let $a = 10^x$ and $b = 10^y$. Our equations become:

$$10a = b/10 + 1$$
$$10b = a + 1,$$

and we are asked to compute a/b. Subtracting the equations gives

$$10a - 10b = b/10 - a \implies 11a = 101b/10,$$

giving an answer of $101/110$.

15. **[9]** Vijay chooses three distinct integers a, b, c from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. If k is the minimum value taken on by the polynomial $a(x - b)(x - c)$ over all real numbers x, and l is the minimum value taken on by the polynomial $a(x - b)(x + c)$ over all real numbers x, compute the maximum possible value of $k - l$.

Proposed by: Preston Bushnell

Answer: $\boxed{990}$

Solution: Quadratics are minimized at the average of their roots, so

$$k = a\left(\frac{b+c}{2} - b\right)\left(\frac{b+c}{2} - c\right) = a\left(\frac{b-c}{2}\right)\left(\frac{c-b}{2}\right) = -\frac{a(b-c)^2}{4}, \text{ and}$$
$$l = a\left(\frac{b-c}{2} - b\right)\left(\frac{b-c}{2} + c\right) = a\left(\frac{-b-c}{2}\right)\left(\frac{b+c}{2}\right) = -\frac{a(b+c)^2}{4}.$$

Therefore,

$$k - l = -\frac{a}{4}\left((b-c)^2 - (b+c)^2\right) = abc$$

Thus, $k - l = abc$ is maximized when a, b, and c are 9, 10, and 11 are some order, so the answer is $9 \cdot 10 \cdot 11 = 990$.

16. **[10]** Given an angle θ, consider the polynomial

$$P(x) = \sin(\theta)x^2 + (\cos(\theta) + \tan(\theta))x + 1.$$

Given that P only has one real root, find all possible values of $\sin(\theta)$.

Proposed by: Eric Shen

Answer: $\boxed{0, \frac{\sqrt{5}-1}{2}}$

Solution: Note that if $\sin(\theta) = 0$, then the polynomial has 1 root. Now assume this is not the case - then the polynomial is a quadratic in x.

Factor the polynomial as $(\tan(\theta)x + 1)(x + \sec(\theta))$. Then the condition is equivalent to $\sec(\theta) = \frac{1}{\tan(\theta)}$, which is equivalent to $\sin(\theta) = \cos^2(\theta) = 1 - \sin^2(\theta)$. Solving now gives $\sin(\theta) = \frac{\sqrt{5}-1}{2}$ as the only solution.

17. **[10]** How many ways are there to color every integer either red or blue such that n and $n + 7$ are the same color for all integers n, and there does not exist an integer k such that k, $k + 1$, and $2k$ are all the same color?

Proposed by: Luke Robitaille

Answer: $\boxed{6}$

Solution: It suffices to color the integers from 0 through 6 and do all arithmetic mod 7. WLOG, say that 0 is red (we'll multiply by 2 in the end). Then 1 must be blue because $(0, 0, 1)$ can't be monochromatic. 2 must be red because $(1, 2, 2)$ can't be monochromatic. Then we have two cases for what 3 is:

Case 1: 3 is red. Then 4 is blue because $(2, 3, 4)$ can't be monochromatic. This makes 5 red because $(4, 5, 1)$ can't be monochromatic. Finally, 6 must be blue because $(6, 0, 5)$ can't be monochromatic. This gives a single consistent coloring for this case.

Case 2: 3 is blue. 4 can't also be blue because this would imply that 5 is red (because of $(4, 5, 1)$) and 6 is red (because of $(3, 4, 6)$), which would make $(6, 0, 5)$ all red. So 4 must be red. Then we have two possibilities: either 5 is red and 6 is blue, or 5 is blue and 6 is red (5 and 6 can't both be red because of $(6, 0, 5)$, and they can't both be blue because of $(5, 6, 3)$). These give two consistent colorings for this case.

Overall, we have three consistent colorings: `RBRRBRB`, `RBRBRRB`, and `RBRBRBR`. Multiply this by 2 because 0 could have been blue, and our answer is 6.

18. **[10]** A regular tetrahedron has a square shadow of area 16 when projected onto a flat surface (light is shone perpendicular onto the plane). Compute the sidelength of the regular tetrahedron.

(For example, the shadow of a sphere with radius 1 onto a flat surface is a disk of radius 1.)

Proposed by: Albert Wang

Answer: $\boxed{4\sqrt{2}}$

Solution: Imagine the shadow of the skeleton of the tetrahedron (i.e. make the entire tetrahedron translucent except for the edges). The diagonals of the square shadow must correspond to a pair of opposite edges of the tetrahedron. Both of these edges must be parallel to the plane – if they weren't, then edges corresponding to the four sides of the square would have to have different lengths. Thus, the length of a diagonal of the square (namely, $4\sqrt{2}$) must be the same as the edge length of the tetrahedron.

19. **[11]** Define the *annoyingness* of a permutation of the first n integers to be the minimum number of copies of the permutation that are needed to be placed next to each other so that the subsequence $1, 2, \ldots, n$ appears. For instance, the annoyingness of $3, 2, 1$ is 3, and the annoyingness of $1, 3, 4, 2$ is 2.

A random permutation of $1, 2, \ldots, 2022$ is selected. Compute the expected value of the annoyingness of this permutation.

Proposed by: Vidur Jasuja

Answer: $\boxed{\frac{2023}{2}}$

Solution: For a given permutation $p_1, \ldots, p_n$, let $f_k(p)$ be the smallest number of copies of p that need to be placed next to each other to have $1, \ldots, k$ appear as a subsequence. We are interested in finding the expectation of $f_n(p)$.

Notice that if k appears before $k + 1$ in p, then $f_k(p) = f_{k+1}(p)$. Otherwise, $f_k(p) + 1 = f_{k+1}(p)$. Since $f_1(p)$ is always 1, this tells us that $f_n(p)$ is equal to 1 plus the number of values of k that exist such that $k + 1$ appears before k. But for any such k, this occurs with probability $1/2$. By linearity of expectation, the answer is $1 + 2021/2 = \frac{2023}{2}$.

20. **[11]** Let $\triangle ABC$ be an isosceles right triangle with $AB = AC = 10$. Let M be the midpoint of BC and N the midpoint of BM. Let AN hit the circumcircle of $\triangle ABC$ again at T. Compute the area of $\triangle TBC$.

Proposed by: Andrew Gu

Answer: $\boxed{30}$

Solution: Note that since quadrilateral $BACT$ is cyclic, we have

$$\angle BTA = \angle BCA = 45^\circ = \angle CBA = \angle CTA.$$

Hence, TA bisects $\angle BTC$, and $\angle BTC = 90^\circ$. By the angle bisector theorem, we then have

$$\frac{BT}{TC} = \frac{BN}{NC} = \frac{1}{3}.$$

By the Pythagorean theorem on right triangles $\triangle TBC$ and $\triangle ABC$, we have

$$10BT^2 = BT^2 + TC^2 = AB^2 + AC^2 = 200,$$

so $BT^2 = 20$. Note that the area of $\triangle TBC$ is

$$\frac{BT \cdot TC}{2} = \frac{3 \cdot BT^2}{2},$$

so our answer is then

$$\frac{3}{2} \cdot BT^2 = \frac{3}{2} \cdot 20 = 30.$$

21. **[11]** Let $P(x)$ be a quadratic polynomial with real coefficients. Suppose that $P(1) = 20$, $P(-1) = 22$, and $P(P(0)) = 400$. Compute the largest possible value of $P(10)$.

Proposed by: Luke Robitaille

Answer: $\boxed{2486}$

Solution: Let $P(x) = ax^2 + bx + c$. The given equations give us:

$$\begin{aligned} a + b + c &= 20 \\ a - b + c &= 22 \end{aligned}$$

Hence $b = -1$, $a + c = 21$, and so the final equation gives us $ac^2 = 400$. Substituting $a = 21 - c$ and solving the cubic in c, we get $c = -4, 5, 20$. Of these, the smallest value $c = -4$ (and hence $P(x) = 25x^2 - x - 4$) ends up giving the largest value of $P(10)$.

22. **[12]** Find the number of pairs of integers (a, b) with $1 \leq a < b \leq 57$ such that a^2 has a smaller remainder than b^2 when divided by 57.

Proposed by: Zixiang Zhou

Answer: $\boxed{738}$

Solution: There are no such pairs when $b = 57$, so we may only consider pairs with $1 \leq a < b \leq 56$. The key idea is that unless $a^2 \bmod 57 = b^2 \bmod 57$, (a, b) can be paired with $(57 - b, 57 - a)$ and exactly one of them satisfies $a^2 \bmod 57 < b^2 \bmod 57$. Hence if X is the number of pairs (a, b) with $1 \leq a < b \leq 56$ and $a^2 \equiv b^2 \pmod{57}$, then the answer is

$$\frac{1}{2}\left(\binom{56}{2} - X\right).$$

To count X, let's first count the number of pairs (a, b) with $1 \le a, b \le 57$ and $a^2 \equiv b^2 \pmod{57}$. By the Chinese Remainder Theorem, the condition is equivalent to $(a-b)(a+b) \equiv 0 \pmod 3$ and $(a-b)(a+b) \equiv 0 \pmod{19}$. There are $2 \cdot 3 - 1 = 5$ pairs of residues modulo 3 where $(a-b)(a+b) \equiv 0 \pmod 3$, namely $(0,0), (1,1), (2,2), (1,2), (2,1)$. Similarly, there are $2 \cdot 19 - 1 = 37$ pairs of residues modulo 19 where $(a-b)(a+b) \equiv 0 \pmod{19}$. By the Chinese Remainder Theorem, each choice of residues modulo 3 for a and b and residues modulo 19 for a and b corresponds to unique residues modulo 57 for a and b. It follows that there are $5 \cdot 37 = 185$ such pairs. To get the value of X, we need to subtract the 57 pairs where $a = b$ and divide by 2 for the pairs with $a > b$, for a value of $X = \frac{1}{2}(185 - 57) = 64$.

Therefore the final answer is

$$\frac{1}{2}\left(\binom{56}{2} - 64\right) = 738.$$

23. **[12]** Let ABC be a triangle with $AB = 2021, AC = 2022$, and $BC = 2023$. Compute the minimum value of $AP + 2BP + 3CP$ over all points P in the plane.

Proposed by: Evan Erickson

Answer: $\boxed{6068}$

Solution 1: The minimizing point is when $P = C$. To prove this, consider placing P at any other point $O \neq C$. Then, by moving P from O to C, the expression changes by

$$(AC - AO) + 2(BC - BO) + 3(CC - CO) < OC + 2OC - 3OC = 0$$

by the triangle inequality. Since this is negative, $P = C$ must be the optimal point. The answer is $2022 + 2 \cdot 2023 + 3 \cdot 0 = 6068$.

Solution 2: We use a physical interpretation. Imagine an object acted upon by forces of magnitudes 1, 2, and 3 towards A, B, and C, respectively. The potential energy of the object at point P in this system is $AP + 2BP + 3CP$. This potential energy is minimized when the object experiences 0 net force; in this case, it occurs when it is exactly at point C (because the pull towards C overpowers the other two forces combined).

24. **[12]** A string consisting of letters `A`, `C`, `G`, and `U` is *untranslatable* if and only if it has no `AUG` as a consecutive substring. For example, `ACUGG` is untranslatable.

Let a_n denote the number of untranslatable strings of length n. It is given that there exists a unique triple of real numbers (x, y, z) such that $a_n = xa_{n-1} + ya_{n-2} + za_{n-3}$ for all integers $n \ge 100$. Compute (x, y, z).

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{(4, 0, -1)}$

Solution: If a sequence is untranslatable, the first $n - 1$ letters must form an untranslatable sequence as well. Therefore, we can count a_n by

- Append any letter to an untranslatable sequence of length $n - 1$, so $4a_{n-1}$ ways.
- Then, subtract with the case when the sequence ends with `AUG`. There are a_{n-3} sequences in this case.

Thus, $a_n = 4a_{n-1} - a_{n-3}$ for all integers $n \ge 3$, so the answer is $(4, 0, -1)$

25. **[13]** In convex quadrilateral $ABCD$ with $AB = 11$ and $CD = 13$, there is a point P for which $\triangle ADP$ and $\triangle BCP$ are congruent equilateral triangles. Compute the side length of these triangles.

Proposed by: Albert Wang

Answer: $\boxed{7}$

Solution:

Evidently $ABCD$ is an isosceles trapezoid with P as its circumcenter. Now, construct isosceles trapezoid $AB'BC$ (that is, BB' is parallel to AC.) Then $AB'PD$ is a rhombus, so $\angle B'CD = \frac{1}{2}\angle B'PD = 60°$ by the inscribed angle theorem. Also, $B'C = 11$ because the quadrilateral $B'APC$ is a $60°$ rotation of $ADPB$ about P. Since $CD = 13$, we use the law of cosines to get that $B'D = 7\sqrt{3}$. Hence $AP = 7$.

26. **[13]** A number is chosen uniformly at random from the set of all positive integers with at least two digits, none of which are repeated. Find the probability that the number is even.

Proposed by: Benjamin Shimabukuro

Answer: $\boxed{\frac{41}{81}}$

Solution: Since the number has at least two digits, all possible combinations of first and last digits have the same number of possibilities, which is $\sum_{i=0}^{8} \frac{8!}{i!}$. Since the first digit cannot be zero, all of the last digits have 8 possible first digits, except for 0, which has 9 possible first digits. Therefore, the probability that the last digit is even is $\frac{9+4\cdot 8}{9+9\cdot 8} = \frac{41}{81}$.

27. **[13]** How many ways are there to cut a 1 by 1 square into 8 congruent polygonal pieces such that all of the interior angles for each piece are either 45 or 90 degrees? Two ways are considered distinct if they require cutting the square in different locations. In particular, rotations and reflections are considered distinct.

Proposed by: Freddie Zhao

Answer: $\boxed{54}$

Solution: First note that only triangles and quadrilaterals are possible.

There are 3 possibilities:

- 1/2 by 1/2 right isosceles triangles
- 1 by 1/8 rectangles
- 1/2 by 1/4 rectangles

The first case has 16 possibilities (there are 2 choices for the orientation of each quadrant). The second case has 2 possibilities (either all horizontal or all vertical).

The third case is the trickiest. Label the quadrants A, B, C, D where A, B are at the top and B, C are on the left. If each rectangle lies completely within a quadrant, there are 16 ways. If rectangles span quadrants A, B but not C or D, there are 4 ways. Similarly, there are 4 ways each for [rectangles spanning B, C but not D, A], [rectangles spanning C, D but not A, B], and [rectangles spanning D, A but not B, C]. Next, if rectangles span both A, B and C, D, there is 1 way, and if rectangles span both B, C and D, A there is 1 way. Finally there are 2 ways for each adjacent pair of quadrants to have a rectangle spanning them. This brings us to $16 + 4 + 4 + 4 + 4 + 1 + 1 + 2 = 36$ ways.

The final answer is $16 + 2 + 36 = 54$.

28. **[15]** Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Pick points Q and R on AC and AB such that $\angle CBQ = \angle BCR = 90°$. There exist two points $P_1 \neq P_2$ in the plane of ABC such that $\triangle P_1QR$, $\triangle P_2QR$, and $\triangle ABC$ are similar (with vertices in order). Compute the sum of the distances from P_1 to BC and P_2 to BC.

Proposed by: Ankit Bisain

Answer: $\boxed{48}$

Solution 1: Let T be the foot of the A-altitude of ABC. Recall that $BT = 5$ and $CT = 9$.

Let T' be the foot of the P-altitude of PQR. Since T' is the midpoint of the possibilities for P, the answer is

$$\sum_P d(P, BC) = 2d(T', BC).$$

Since T' splits QR in a $5 : 9$ ratio, we have

$$d(T', BC) = \frac{9d(Q, BC) + 5d(R, BC)}{14}.$$

By similar triangles, $d(Q, BC) = QB = 12 \cdot \frac{14}{9}$, and similar for $d(R, BC)$, giving $d(T', BC) = 24$, and an answer of 48.

Solution 2: As in the previous solution, let T be the foot from A to BC, let T' be the foot from P to QR. and recall that

$$d(P_1, BC) + d(P_2, BC) = 2d(T', BC).$$

Now, notice that since $\triangle PQR \sim \triangle ABC$, we have $QT' : T'R = BT : TC$, so $TT' \parallel BQ \parallel CR$, implying that $A \in TT'$.

However, we recall a well-known fact that A is the midpoint of TT' (can be proven by simple similar triangles). Thus, $d(T', BC)$ is equal to two times the altitude from A to BC. Hence, the answer is four times the altitude from A to BC, which is 48.

29. **[15]** Consider the set S of all complex numbers z with nonnegative real and imaginary part such that

$$|z^2 + 2| \leq |z|.$$

Across all $z \in S$, compute the minimum possible value of $\tan\theta$, where θ is the angle formed between z and the real axis.

Proposed by: Vidur Jasuja

Answer: $\boxed{\sqrt{7}}$

Solution: Let $z = a + bi$. Then,

$$z^2 + 2 = (a^2 - b^2 + 2)^2 + 2ab \cdot i.$$

Recall the identity $(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2$, so we have

$$|z^2 + 2|^2 = (a^2 + b^2)^2 + 4(a^2 - b^2) + 4$$

Thus, $z \in S$ if and only if $(a^2 + b^2)^2 + 4(a^2 - b^2) + 4 \leq a^2 + b^2$.

Suppose $(a^2 - b^2) = c(a^2 + b^2)$. Note that $|c| < 1$. Let $a^2 + b^2 = x$. Then, $x^2 + (4c - 1)x + 4 \leq 0$. This inequality must have real solutions, forcing $-4 < 4c - 1 < 4$, implying that $-1 \leq c \leq -\frac{3}{4}$. The smaller the magnitude of c, the smaller the ratio $\frac{b^2}{a^2}$, so then $c = -\frac{3}{4}$. This gives us that $a^2 : b^2 = 1 : 7$, so then $\tan\theta \geq \sqrt{7}$. This value of c achieves exactly one solution, at $x = 2$, in particular $z = \frac{1}{2} + \frac{i\sqrt{7}}{2}$.

30. **[15]** Let ABC be a triangle with $AB = 8$, $AC = 12$, and $BC = 5$. Let M be the second intersection of the internal angle bisector of $\angle BAC$ with the circumcircle of ABC. Let ω be the circle centered at M tangent to AB and AC. The tangents to ω from B and C, other than AB and AC respectively, intersect at a point D. Compute AD.

Proposed by: Eric Shen

Answer: $\boxed{16}$

Solution: Redefine D as the reflection of A across the perpendicular bisector l of BC. We prove that DB and DC are both tangent to ω, and hence the two definitions of D align. Indeed, this follows by symmetry; we have that $\angle CBM = \angle CAM = \angle BAM = \angle BCM$, so $BM = CM$ and so ω is centered on and hence symmetric across l. Hence reflecting BAC across l, we get that DB, DC are also tangent to ω, as desired.

Hence we have by Ptolemy that $5AD = 12^2 - 8^2$, so thus $AD = 16$.

31. **[17]** Given positive integers $a_1, a_2, \ldots, a_{2023}$ such that

$$a_k = \sum_{i=1}^{2023} |a_k - a_i|$$

for all $1 \leq k \leq 2023$, find the minimum possible value of $a_1 + a_2 + \cdots + a_{2023}$.

Proposed by: Maxim Li

Answer: $\boxed{2046264}$

Solution: Without loss of generality, let $a_1 \leq a_2 \leq \cdots \leq a_{2023}$. Then, note that

$$\begin{aligned} a_{k+1} - a_k &= \sum_{i=1}^{2023} |a_{k+1} - a_i| - |a_k - a_i| \\ &= k(a_{k+1} - a_k) - (2023 - k)(a_{k+1} - a_k) \\ &= (2k - 2023)(a_{k+1} - a_k). \end{aligned}$$

Thus, $a_{k+1} = a_k$ unless $k = 1012$, so $a_1 = a_2 = \cdots = a_{1012}$ and $a_{1013} = \cdots = a_{2023}$, and we can check that that they must be in a ratio of $1011 : 1012$. Thus, $a_1, \ldots, a_{2023}$ must consist of 1012 copies of $1011c$, and 1011 copies of $1012c$ for some c, so for everything to be a positive integer, we need $c \geq 1$. This gives us the final answer of $1011 \cdot 1012 + 1012 \cdot 1011 = 2046264$.

32. **[17]** Suppose point P is inside triangle ABC. Let AP, BP, and CP intersect sides BC, CA, and AB at points D, E, and F, respectively. Suppose $\angle APB = \angle BPC = \angle CPA$, $PD = \frac{1}{4}$, $PE = \frac{1}{5}$, and $PF = \frac{1}{7}$. Compute $AP + BP + CP$.

Proposed by: Rishabh Das

Answer: $\boxed{\frac{19}{12}}$

Solution:

The key is the following lemma:

Lemma: If $\angle X = 120°$ in $\triangle XYZ$, and the bisector of X intersects YZ at T, then

$$\frac{1}{XY} + \frac{1}{XZ} = \frac{1}{XT}.$$

Proof of the Lemma. Construct point W on XY such that $\triangle XWT$ is equilateral. We also have $TW \parallel XZ$. Thus, by similar triangles,

$$\frac{XT}{XZ} = \frac{YT}{YX} = 1 - \frac{XT}{XY},$$

implying the conclusion. $\square$

Now we can write

$$\begin{aligned} \frac{1}{PB} + \frac{1}{PC} &= 4, \\ \frac{1}{PC} + \frac{1}{PA} &= 5, \text{ and} \\ \frac{1}{PA} + \frac{1}{PB} &= 7. \end{aligned}$$

From here we can solve to obtain $\frac{1}{PA} = 4$, $\frac{1}{PB} = 3$, $\frac{1}{PC} = 1$, making the answer $\frac{19}{12}$.

33. **[17]** A group of 101 Dalmathians participate in an election, where they each vote independently on either candidate A or B with equal probability. If X Dalmathians voted for the winning candidate, the expected value of X^2 can be expressed as $\frac{a}{b}$ for positive integers a, b with $\gcd(a, b) = 1$. Find the unique positive integer $k \leq 103$ such that $103 \mid a - bk$.

Proposed by: William Wang

Answer: $\boxed{51}$

Solution: Claim: with 101 replaced with $2k+1$, the expectation of X^2 is

$$\frac{\binom{2k}{k}}{2^{2k+1}}(2k+1)^2 + \frac{(2k+1)(2k+2)}{4}.$$

The answer is this value taken modulo 103, which can be calculated by noting that the integers modulo 103 form a finite field. Note that the multiplicative inverse of 4 is 26, the multiplicative inverse of 2^{101} is 2 by Fermat's little theorem, and the multiplicative inverse of 102! is 102 by Wilson's theorem.

Now we will justify the Claim. Let I_i be the indicator random variable of the i-th Dalmathian voting for the winning candidate ($I_i = 1$ if i votes for the winning candidate, and $I_i = 0$ otherwise). Then we want to find

$$\mathbb{E}[(I_1 + \cdots + I_{2k+1})^2].$$

By symmetry and linearity, this is

$$(2k+1)\mathbb{E}[I_1^2] + (2k+1)(2k)\mathbb{E}[I_1 I_2].$$

Now, we note that $\mathbb{E}[I_1^2] = \mathbb{E}[I_1]$ is just the probability that Dalmathian 1 votes for the winning candidate. WLOG, say that they vote for A. Then we want to find the probability that at least k of the remaining $2k$ Dalmathians also vote for A. By symmetry, this is equal to the probability that exactly k vote for A, plus half of the remaining probability. This is:

$$\frac{1}{2} + \frac{\binom{2k}{k}}{2^{2k+1}}.$$

Next, we must calculate $\mathbb{E}[I_1 I_2]$. In order for $I_1 I_2$ to be 1, they must Dalmathians vote for the same candidate (1/2 chance), and then this candidate has to win (at least $k-1$ out of the remaining $2k-1$ Dalmathians vote for that candidate). Overall, this occurs with probability

$$\frac{1}{2}\left(\frac{1}{2} + \frac{\binom{2k-1}{k-1}}{2^{2k-1}}\right).$$

Now when we add the two terms together, we get

$$\left(\frac{1}{2}+\frac{\binom{2k}{k}}{2^{2k+1}}\right)(2k+1)+(2k+1)(2k)\left(\frac{1}{4}+\frac{\binom{2k-1}{k-1}}{2^{2k}}\right).$$

With some simplification, you get the expression in the Claim.

34. **[20]** A random binary string of length 1000 is chosen. Let L be the expected length of its longest (contiguous) palindromic substring. Estimate L.

An estimate of E will receive $\lfloor 20\min(\frac{E}{L},\frac{L}{E})^{10}\rfloor$ points.

Proposed by: Gabriel Wu

Answer: $\boxed{L \approx 23.120}$

Solution: The probability that there exists a palindromic substring of length $2n+1$ is approximately $2^{-n}\cdot 1000$. Thus, we can expect to often see a length 21 palindrome, and sometimes longer ones. This leads to a guess a bit above 21.

L was approximated with 10^7 simulations (the answer is given with a standard deviation of about 10^{-3}).

35. **[20]** For each $i \in \{1,\ldots,10\}$, a_i is chosen independently and uniformly at random from $[0,i^2]$. Let P be the probability that $a_1 < a_2 < \cdots < a_{10}$. Estimate P.

An estimate of E will earn $\lfloor 20\min(\frac{E}{P},\frac{P}{E})\rfloor$ points.

Proposed by: Gabriel Wu

Answer: $\boxed{P \approx 0.003679}$

Solution: The probability that $a_2 > a_1$ is $7/8$. The probability that $a_3 > a_2$ is $7/9$. The probability that $a_4 > a_3$ is $23/32$. The probability that $a_5 > a_4$ is $17/25$. The probability that $a_6 > a_5$ is $47/72$. The probability that $a_7 > a_6$ is $31/49$. The probability that $a_8 > a_7$ is $79/128$. The probability that $a_9 > a_8$ is $49/81$.The probability that $a_{10} > a_9$ is $119/200$.

Assuming all of these events are independent, you can multiply the probabilities together to get a probability of around 0.05. However, the true answer should be less because, conditioned on the realization of $a_1 < a_2 < \cdots < a_k$, the value of a_k is on average large for its interval. This makes $a_k < a_{k+1}$ less likely. Although this effect is small, when compounded over 9 inequalities we can estimate that it causes the answer to be about $1/10$ of the fully independent case.

P was approximated with 10^9 simulations (the answer is given with a standard deviation of about 2×10^{-6}).

36. **[20]** Consider all questions on this year's contest that ask for a single real-valued answer (excluding this one). Let M be the median of these answers. Estimate M.

An estimate of E will earn $\lfloor 20\min(\frac{E}{M},\frac{M}{E})^4\rfloor$ points.

Proposed by: Gabriel Wu, Jerry Liang

Answer: $\boxed{M = 8+8\sqrt[4]{3} \approx 18.5285921}$

Solution: Looking back to the answers of previous problems in the round (or other rounds) can give you to a rough estimate.