# HMMT November

**November 12, 2022**

## Theme Round

1. Alice and Bob are playing in an eight-player single-elimination rock-paper-scissors tournament. In the first round, all players are paired up randomly to play a match. Each round after that, the winners of the previous round are paired up randomly. After three rounds, the last remaining player is considered the champion. Ties are broken with a coin flip. Given that Alice always plays rock, Bob always plays paper, and everyone else always plays scissors, what is the probability that Alice is crowned champion? Note that rock beats scissors, scissors beats paper, and paper beats rock.

2. Alice is thinking of a positive real number $x$, and Bob is thinking of a positive real number $y$. Given that $x^{\sqrt{y}} = 27$ and $(\sqrt{x})^y = 9$, compute $xy$.

3. Alice is bored in class, so she thinks of a positive integer. Every second after that, she subtracts from her current number its smallest prime divisor, possibly itself. After 2022 seconds, she realizes that her number is prime. Find the sum of all possible values of her initial number.

4. Alice and Bob stand atop two different towers in the Arctic. Both towers are a positive integer number of meters tall and are a positive (not necessarily integer) distance away from each other. One night, the sea between them has frozen completely into reflective ice. Alice shines her flashlight directly at the top of Bob's tower, and Bob shines his flashlight at the top of Alice's tower by first reflecting it off the ice. The light from Alice's tower travels 16 meters to get to Bob's tower, while the light from Bob's tower travels 26 meters to get to Alice's tower. Assuming that the lights are both shone from exactly the top of their respective towers, what are the possibilities for the height of Alice's tower?

5. Alice is once again very bored in class. On a whim, she chooses three primes $p$, $q$, $r$ independently and uniformly at random from the set of primes of at most 30. She then calculates the roots of $px^2+qx+r$. What is the probability that at least one of her roots is an integer?

6. A regular octagon is inscribed in a circle of radius 2. Alice and Bob play a game in which they take turns claiming vertices of the octagon, with Alice going first. A player wins as soon as they have selected three points that form a right angle. If all points are selected without either player winning, the game ends in a draw. Given that both players play optimally, find all possible areas of the convex polygon formed by Alice's points at the end of the game.

7. Alice and Bob are playing in the forest. They have six sticks of length $1, 2, 3, 4, 5, 6$ inches. Somehow, they have managed to arrange these sticks, such that they form the sides of an equiangular hexagon. Compute the sum of all possible values of the area of this hexagon.

8. Alice thinks of four positive integers $a \leq b \leq c \leq d$ satisfying $\{ab+cd, ac+bd, ad+bc\} = \{40, 70, 100\}$. What are all the possible tuples $(a, b, c, d)$ that Alice could be thinking of?

9. Alice and Bob play the following "point guessing game." First, Alice marks an equilateral triangle $ABC$ and a point $D$ on segment $BC$ satisfying $BD = 3$ and $CD = 5$. Then, Alice chooses a point $P$ on line $AD$ and challenges Bob to mark a point $Q \neq P$ on line $AD$ such that $\frac{BQ}{QC} = \frac{BP}{PC}$. Alice wins if and only if Bob is unable to choose such a point. If Alice wins, what are the possible values of $\frac{BP}{PC}$ for the $P$ she chose?

10. There are 21 competitors with distinct skill levels numbered $1, 2, \ldots, 21$. They participate in a ping-pong tournament as follows. First, a random competitor is chosen to be "active", while the rest are "inactive." Every round, a random inactive competitor is chosen to play against the current active one. The player with the higher skill will win and become (or remain) active, while the loser will be eliminated from the tournament. The tournament lasts for 20 rounds, after which there will only be one player remaining. Alice is the competitor with skill 11. What is the expected number of games that she will get to play?