HMMT November 2022

November 12, 2022

Theme Round

1. Alice and Bob are playing in an eight-player single-elimination rock-paper-scissors tournament. In the first round, all players are paired up randomly to play a match. Each round after that, the winners of the previous round are paired up randomly. After three rounds, the last remaining player is considered the champion. Ties are broken with a coin flip. Given that Alice always plays rock, Bob always plays paper, and everyone else always plays scissors, what is the probability that Alice is crowned champion? Note that rock beats scissors, scissors beats paper, and paper beats rock.

   Proposed by: Reagan Choi

   Answer: $\boxed{\frac{6}{7}}$

   Solution: Alice's opponent is chosen randomly in the first round. If Alice's first opponent is Bob, then she will lose immediately to him. Otherwise, Bob will not face Alice in the first round. This means he faces someone who plays scissors, so Bob will lose in the first round. Also, this means Alice will never face Bob; and since all other six possible opponents will play scissors, Alice's rock will beat all of them, so she will win the tournament. Hence, since 6 of the 7 first-round opponents lead to wins, the probability that Alice wins is $\frac{6}{7}$.

2. Alice is thinking of a positive real number x, and Bob is thinking of a positive real number y. Given that $x^{\sqrt{y}} = 27$ and $(\sqrt{x})^y = 9$, compute xy.

   Proposed by: Sean Li

   Answer: $\boxed{16\sqrt[4]{3}}$

   Solution: Note that
$$27^{\sqrt{y}} = (x^{\sqrt{y}})^{\sqrt{y}} = x^y = (\sqrt{x})^{2y} = 81,$$
   so $\sqrt{y} = 4/3$ or $y = 16/9$. It follows that $x^{4/3} = 27$ or $x = 9\sqrt[4]{3}$. The final answer is $9\sqrt[4]{3} \cdot 16/9 = 16\sqrt[4]{3}$.

3. Alice is bored in class, so she thinks of a positive integer. Every second after that, she subtracts from her current number its smallest prime divisor, possibly itself. After 2022 seconds, she realizes that her number is prime. Find the sum of all possible values of her initial number.

   Proposed by: Maxim Li

   Answer: $\boxed{8093}$

   Solution: Let a_k denote Alice's number after k seconds, and let p_k be the smallest prime divisor of a_k. We are given that a_{2022} is prime, and want to find a_0.

   If a_0 is even, then $a_{n+1} = a_n - 2$, since every a_n is even. Then we need $a_{2022} = 2$, so $a_0 = 4046$.

   If a_0 is odd, then $a_1 = a_0 - p_0$ is even, so by similar logic to the even case, $a_1 = 4044$. Then since $p_0 | a_0 - p_0$ and $4044 = 4 \cdot 3 \cdot 337$, we must have $p_0 = 3$ or 337. But if $p_0 = 337$, $a_0 = 12 \cdot 337 + 337 = 13 \cdot 337$, so 337 is not the smallest prime divisor of a_0. Thus, we need $p_0 = 3$, so $a_0 = 4047$, which works.

   Thus, the final answer is $4046 + 4047 = 8093$.

4. Alice and Bob stand atop two different towers in the Arctic. Both towers are a positive integer number of meters tall and are a positive (not necessarily integer) distance away from each other. One night, the sea between them has frozen completely into reflective ice. Alice shines her flashlight directly at the top of Bob's tower, and Bob shines his flashlight at the top of Alice's tower by first reflecting it

off the ice. The light from Alice's tower travels 16 meters to get to Bob's tower, while the light from Bob's tower travels 26 meters to get to Alice's tower. Assuming that the lights are both shone from exactly the top of their respective towers, what are the possibilities for the height of Alice's tower?

Proposed by: Eric Shen

Answer: $\boxed{7, 15}$

Solution:

Let Alice's tower be of a height a, and Bob's tower a height b. Reflect the diagram over the ice to obtain an isosceles trapezoid. Then we get that by Ptolemy's Theorem, $4ab = 26^2 - 16^2 = 4 \cdot 105$, thus $ab = 105$. Hence $a \in \{1, 3, 5, 7, 15, 21, 35, 105\}$. But $\max(a, b) \leq 26 + 16 = 42$ by the Triangle inequality, so thus $a \notin \{1, 105\}$. Also, $3, 5, 21$, and 35 don't work because $a + b < 26$ and $|a - b| < 16$.

5. Alice is once again very bored in class. On a whim, she chooses three primes p, q, r independently and uniformly at random from the set of primes of at most 30. She then calculates the roots of $px^2 + qx + r$. What is the probability that at least one of her roots is an integer?

Proposed by: Eric Shen

Answer: $\boxed{\frac{3}{200}}$

Solution: Since all of the coefficients are positive, any root x must be negative. Moreover, by the rational root theorem, in order for x to be an integer we must have either $x = -1$ or $x = -r$. So we must have either $pr^2 - qr + r = 0 \iff pr = q - 1$ or $p - q + r = 0$. Neither of these cases are possible if all three primes are odd, so we know so we know that one of the primes is even, hence equal to 2. After this we can do a casework check; the valid triples of (p, q, r) are $(2, 5, 3)$, $(2, 7, 5)$, $(2, 13, 11)$, $(2, 19, 17)$, $(2, 5, 2)$, $(2, 7, 3)$, $(2, 11, 5)$, $(2, 23, 11)$, allowing for p and r to be swapped. This leads to 15 valid triples out of 1000 (there are 10 primes less than 30).

6. A regular octagon is inscribed in a circle of radius 2. Alice and Bob play a game in which they take turns claiming vertices of the octagon, with Alice going first. A player wins as soon as they have selected three points that form a right angle. If all points are selected without either player winning, the game ends in a draw. Given that both players play optimally, find all possible areas of the convex polygon formed by Alice's points at the end of the game.

Proposed by: Rishabh Das

Answer: $\boxed{2\sqrt{2}, 4 + 2\sqrt{2}}$

Solution: A player ends up with a right angle iff they own two diametrically opposed vertices. Under optimal play, the game ends in a draw: on each of Bob's turns he is forced to choose the diametrically opposed vertex of Alice's most recent choice, making it impossible for either player to win. At the end, the two possibilities are Alice's points forming the figure in red or the figure in blue (and rotations of these shapes). The area of the red quadrilateral is $3[\triangle OAB] - [\triangle OAD] = 2\sqrt{2}$ (this can be computed using the $\frac{1}{2}ab\sin\theta$ formula for the area of a triangle). The area of the blue quadrilateral can be calculated similarly by decomposing it into four triangles sharing O as a vertex, giving an area of $4 + 2\sqrt{2}$.

7. Alice and Bob are playing in the forest. They have six sticks of length $1, 2, 3, 4, 5, 6$ inches. Somehow, they have managed to arrange these sticks, such that they form the sides of an equiangular hexagon. Compute the sum of all possible values of the area of this hexagon.

   Proposed by: Vidur Jasuja

   Answer: $\boxed{33\sqrt{3}}$

   Solution: Let the side lengths, in counterclockwise order, be a, b, c, d, e, f. Place the hexagon on the coordinate plane with edge a parallel to the x-axis and the intersection between edge a and edge f at the origin (oriented so that edge b lies in the first quadrant). If you travel along all six sides of the hexagon starting from the origin, we get that the final x coordinate must be $a+b/2-c/2-d-e/2+f/2=0$ by vector addition. Identical arguments tell us that we must also have $b+c/2-d/2-e-f/2+a/2=0$ and $c+d/2-e/2-f-a/2+b/2=0$.

   Combining these linear equations tells us that $a-d=e-b=c-f$. This is a necessary and sufficient condition for the side lengths to form an equiangular hexagon. WLOG say that $a=1$ and $b<f$ (otherwise, you can rotate/reflect it to get it to this case).

   Thus, we must either have $(a,b,c,d,e,f)=(1,5,3,4,2,6)$ or $(1,4,5,2,3,6)$. Calculating the areas of these two cases gets either $67\sqrt{3}/4$ or $65\sqrt{3}/4$, for a sum of $33\sqrt{3}$.

8. Alice thinks of four positive integers $a \le b \le c \le d$ satisfying $\{ab+cd, ac+bd, ad+bc\} = \{40, 70, 100\}$. What are all the possible tuples (a, b, c, d) that Alice could be thinking of?

   Proposed by: Vidur Jasuja

   Answer: $\boxed{(1,4,6,16)}$

   Solution: Since $ab \cdot cd = ac \cdot bd = ad \cdot bc$, the largest sum among $ab+cd, ac+bd, ad+bc$ will be the one with the largest difference between the two quantities, so $ab+cd=100, ac+bd=70, ad+bc=40$.

   Consider the sum of each pair of equations, which gives $(a+b)(c+d)=110, (a+c)(b+d)=140, (a+d)(b+c)=170$. Since each of these are pairs summing to $S=a+b+c+d$, by looking at the discriminant of the quadratics with roots $a+b$ and $c+d$, $a+c$ and $b+d$, and $a+d$ and $b+c$, we have that $S^2-680, S^2-560, S^2-440$ must be perfect squares.

   Therefore, we need to find all arithmetic progressions of three squares with common difference 120. The equation $x^2-y^2=120$ has $(31,29), (17,13), (13,7), (11,1)$ as a solution, and so the only possibility is $49, 169, 289$. This implies that $S^2=729 \implies S=27$.

   We now need to verify this works. Note that this implies $\{a+b, c+d\} = \{5, 22\}, \{a+c, b+d\} = \{7, 20\}, \{a+d, b+c\} = \{10, 17\}$. Therefore, $a+b=5$, $a+c=7$. This means that $b+c$ is even, so $b+c=10$. This gives us $(a,b,c,d)=(1,4,6,16)$ is the only possibility, as desired.

9. Alice and Bob play the following "point guessing game." First, Alice marks an equilateral triangle ABC and a point D on segment BC satisfying $BD=3$ and $CD=5$. Then, Alice chooses a point P on line AD and challenges Bob to mark a point $Q \neq P$ on line AD such that $\frac{BQ}{QC}=\frac{BP}{PC}$. Alice wins if and only if Bob is unable to choose such a point. If Alice wins, what are the possible values of $\frac{BP}{PC}$ for the P she chose?

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{\frac{\sqrt{3}}{3}, 1, \frac{3\sqrt{3}}{5}}$

   Solution: First, if $P=A$ then clearly Bob cannot choose a Q. So we can have $BP:PC=1$.

   Otherwise, we need AP to be tangent to the Apollonius Circle. The key claim is that $AB=AC=AP$. To see why, simply note that since B and C are inverses with respect to the Apollonius Circle, we get that $\odot(A, AB)$ and the Apollonius Circle are orthogonal. This gives the claim.

Finding answer is easy. Let M be the midpoint of BC, and let T be the center of that Apollonius Circle. We easily compute $AD = 7$, so we have two cases.

- If P lies on $\overrightarrow{AD}$, then $DP = 1$. Since $\triangle DPT \sim \triangle ADM$, we get that $TD = 7$. Thus, $\frac{BP}{PC} = \sqrt{\frac{BT}{TC}} = \sqrt{\frac{4}{12}} = \frac{1}{\sqrt{3}}$.
- Now, note that the two ratios must have product $BD/DC = 3/5$ by the Ratio lemma. So the other ratio must be $\frac{3\sqrt{3}}{5}$.

Therefore, the solution set is $\left\{\frac{1}{\sqrt{3}}, 1, \frac{3\sqrt{3}}{5}\right\}$.

10. There are 21 competitors with distinct skill levels numbered $1, 2, \ldots, 21$. They participate in a ping-pong tournament as follows. First, a random competitor is chosen to be "active", while the rest are "inactive." Every round, a random inactive competitor is chosen to play against the current active one. The player with the higher skill will win and become (or remain) active, while the loser will be eliminated from the tournament. The tournament lasts for 20 rounds, after which there will only be one player remaining. Alice is the competitor with skill 11. What is the expected number of games that she will get to play?

Proposed by: Zixiang Zhou

Answer: $\boxed{\frac{47}{42}}$

Solution 1: Insert a player with skill level 0, who will be the first active player (and lose their first game).

If Alice plays after any of the players with skill level $12, 13, \ldots, 21$, which happens with probability $\frac{10}{11}$, then she will play exactly 1 game.

If Alice is the first of the players with skill level $11, 12, \ldots, 21$, which happens with probability $\frac{1}{11}$, then there are an expected $\frac{10}{12}$ players between her and someone better than her. Thus, she plays an expected $2 + \frac{10}{12} = \frac{17}{6}$ games.

Alice will only play the player with skill 0 if she is the first of all other players, which happens with probability $\frac{1}{21}$.

The final answer is

$$\frac{10}{11} \cdot 1 + \frac{1}{11} \cdot \frac{17}{6} - \frac{1}{21} = \frac{47}{42}.$$

Solution 2: Replace 21 by n and 11 by k. The general formula is $\frac{n+1}{(n-k+1)(n-k+2)} + 1 - \frac{1}{n} - [k = n]$.

The problem is roughly equivalent to picking a random permutation of $1, \ldots, n$ and asking the expected number of prefix maximums that are equal to k. For the first m elements, the probability is equal to

$$\begin{aligned}
P(\text{max of first } m = k) &= P(\text{max of first } m \leq k) - P(\text{max of first } m \leq k-1) \\
&= \frac{\binom{k}{m} \cdot m! \cdot (n-m)!}{n!} - \frac{\binom{k-1}{m} \cdot m! \cdot (n-m)!}{n!} \\
&= \frac{\binom{k}{m}}{\binom{n}{m}} - \frac{\binom{k-1}{m}}{\binom{n}{m}} \\
&= \frac{\binom{k-1}{m-1}}{\binom{n}{m}}
\end{aligned}$$

so

$$
\begin{aligned}
E[\text{prefix max} = k] &= \sum_{m=1}^{k} \frac{\binom{k-1}{m-1}}{\binom{n}{m}} \\
&= \sum_{m=1}^{k} \frac{(k-1)!m!(n-m)!}{(k-m)!(m-1)!n!} \\
&= \frac{(k-1)!}{n!} \sum_{m=1}^{k} \frac{m(n-m)!}{(k-m)!} \\
&= \frac{(k-1)!(n-k)!}{n!} \sum_{m=1}^{k} m \binom{n-m}{n-k}
\end{aligned}
$$

Now a combinatorial interpretation of the sum is having n balls in a row, choosing a divider between them, and choosing 1 ball on the left side of the divider and $n-k$ balls on the right side of the divider (m corresponds to the number of balls left of the divider). This is equal to choosing $n-k+2$ objects among $n+1$ objects and letting the second smallest one correspond to the divider, which is $\binom{n+1}{n-k+2}$. Therefore the answer is

$$
\frac{(k-1)!(n-k)!}{n!} \cdot \frac{(n+1)!}{(n-k+2)!(k-1)!} = \frac{n+1}{(n-k+1)(n-k+2)}.
$$

We need to do some more careful counting to address the game lost by person k and to subtract 1 game for the event that person k is the first person in the permutation. This yields the $1 - \frac{1}{n} - [k = n]$ term.

The numbers 21 and 11 are chosen so that the answer simplifies nicely.