HMMT February 2023

February 18, 2023

Combinatorics Round

1. There are 800 marbles in a bag. Each marble is colored with one of 100 colors, and there are eight marbles of each color. Anna draws one marble at a time from the bag, without replacement, until she gets eight marbles of the same color, and then she immediately stops.

   Suppose Anna has not stopped after drawing 699 marbles. Compute the probability that she stops immediately after drawing the 700th marble.

2. Compute the number of ways to tile a 3×5 rectangle with one 1×1 tile, one 1×2 tile, one 1×3 tile, one 1×4 tile, and one 1×5 tile. (The tiles can be rotated, and tilings that differ by rotation or reflection are considered distinct.)

3. Richard starts with the string $\mathsf{HHMMMMTT}$. A *move* consists of replacing an instance of HM with MH, replacing an instance of MT with TM, or replacing an instance of TH with HT. Compute the number of possible strings he can end up with after performing zero or more moves.

4. The cells of a 5×5 grid are each colored red, white, or blue. Sam starts at the bottom-left cell of the grid and walks to the top-right cell by taking steps one cell either up or to the right. Thus, he passes through 9 cells on his path, including the start and end cells. Compute the number of colorings for which Sam is guaranteed to pass through a total of exactly 3 red cells, exactly 3 white cells, and exactly 3 blue cells no matter which route he takes.

5. Elbert and Yaiza each draw 10 cards from a 20-card deck with cards numbered $1, 2, 3, \ldots, 20$. Then, starting with the player with the card numbered 1, the players take turns placing down the lowest-numbered card from their hand that is greater than every card previously placed. When a player cannot place a card, they lose and the game ends.

   Given that Yaiza lost and 5 cards were placed in total, compute the number of ways the cards could have been initially distributed. (The order of cards in a player's hand does not matter.)

6. Each cell of a 3×3 grid is labeled with a digit in the set $\{1, 2, 3, 4, 5\}$. Then, the maximum entry in each row and each column is recorded. Compute the number of labelings for which every digit from 1 to 5 is recorded at least once.

7. Svitlana writes the number 147 on a blackboard. Then, at any point, if the number on the blackboard is n, she can perform one of the following three operations:

   - if n is even, she can replace n with $\frac{n}{2}$;
   - if n is odd, she can replace n with $\frac{n+255}{2}$; and
   - if $n \geq 64$, she can replace n with $n - 64$.

   Compute the number of possible values that Svitlana can obtain by doing zero or more operations.

8. A random permutation $a = (a_1, a_2, \ldots, a_{40})$ of $(1, 2, \ldots, 40)$ is chosen, with all permutations being equally likely. William writes down a 20×20 grid of numbers b_{ij} such that $b_{ij} = \max(a_i, a_{j+20})$ for all $1 \leq i, j \leq 20$, but then forgets the original permutation a. Compute the probability that, given the values of b_{ij} alone, there are exactly 2 permutations a consistent with the grid.

9. There are 100 people standing in a line from left to right. Half of them are randomly chosen to face right (with all $\binom{100}{50}$ possible choices being equally likely), and the others face left. Then, while there is a pair of people who are facing each other and have no one between them, the leftmost such pair leaves the line. Compute the expected number of people remaining once this process terminates.

10. Let $x_0 = x_{101} = 0$. The numbers $x_1, x_2, \ldots, x_{100}$ are chosen at random from the interval $[0, 1]$ uniformly and independently. Compute the probability that $2x_i \geq x_{i-1} + x_{i+1}$ for all $i = 1, 2, \ldots, 100$.