HMMT February 2023

February 18, 2023

Guts Round

1. **[10]** Suppose a and b are positive integers such that $a^b = 2^{2023}$. Compute the smallest possible value of b^a.

   Proposed by: Milan Haiman

   Answer: $\boxed{1}$

   Solution: By taking $a = 2^{2023}$ and $b = 1$, we get $b^a = 1$, which is clearly the minimum.

2. **[10]** Let n be a positive integer, and let s be the sum of the digits of the base-four representation of $2^n - 1$. If $s = 2023$ (in base ten), compute n (in base ten).

   Proposed by: Dongyao Jiang

   Answer: $\boxed{1349}$

   Solution: Every power of 2 is either represented in base 4 as $100\ldots00_4$ or $200..00_4$ with some number of zeros. That means every positive integer in the form $2^n - 1$ is either represented in base 4 as $333\ldots33_4$ or $133\ldots33$ for some number threes. Note that $2023 = 2022 + 1 = 674 \cdot 3 + 1$, meaning $2^n - 1$ must be $133\ldots333_4$ with 674 threes. Converting this to base 2 results in

   $$133\ldots33_4 = 200\ldots00_4 - 1 = 2 \cdot 4^{674} - 1 = 2^{1349} - 1$$

   for an answer of 1349.

3. **[10]** Let $ABCD$ be a convex quadrilateral such that $\angle ABD = \angle BCD = 90^\circ$, and let M be the midpoint of segment BD. Suppose that $CM = 2$ and $AM = 3$. Compute AD.

   Proposed by: Luke Robitaille, Milan Haiman

   Answer: $\boxed{\sqrt{21}}$

   Solution: Since triangle BCD is a right triangle, we have $CM = BM = DM = 2$. With $AM = 3$ and $\angle ABM = 90^\circ$, we get $AB = \sqrt{5}$. Now

   $$AD^2 = AB^2 + BD^2 = 5 + 16 = 21,$$

   so $AD = \sqrt{21}$.

4. **[10]** A *standard n-sided die* has n sides labeled 1 to n. Luis, Luke, and Sean play a game in which they roll a fair standard 4-sided die, a fair standard 6-sided die, and a fair standard 8-sided die, respectively. They lose the game if Luis's roll is less than Luke's roll, and Luke's roll is less than Sean's roll. Compute the probability that they lose the game.

   Proposed by: Isabella Quan

   Answer: $\boxed{\frac{1}{4}}$

   Solution: We perform casework on Luke's roll. If Luke rolls n, with $2 \leq n \leq 5$, then the probability Luis rolls less than Luke is $\frac{n-1}{4}$, and the probability Sean rolls more than Luke is $\frac{8-n}{8}$. If Luke rolls 6 then Luis will definitely roll less than Luke, and Sean rolls more than Luke with probability $\frac{2}{8} = \frac{1}{4}$. (If Luke rolls 1 then they cannot lose the game.) Thus, the probability they lose is

   $$\frac{1}{6}\left(\frac{1}{4} \cdot \frac{6}{8} + \frac{2}{4} \cdot \frac{5}{8} + \frac{3}{4} \cdot \frac{4}{8} + \frac{4}{4} \cdot \frac{3}{8} + \frac{1}{4}\right) = \frac{48}{6 \cdot 4 \cdot 8} = \frac{1}{4}.$$

5. **[11]** If a and b are positive real numbers such that $a \cdot 2^b = 8$ and $a^b = 2$, compute $a^{\log_2 a} 2^{b^2}$.

Proposed by: Daniel Hong

Answer: $\boxed{128}$

Solution: Taking $\log_2$ of both equations gives $\log_2 a + b = 3$ and $b \log_2 a = 1$. We wish to find $a^{\log_2 a} 2^{b^2}$; taking $\log_2$ of that gives $(\log_2 a)^2 + b^2$, which is equal to $(\log_2 a + b)^2 - 2b \log_2 a = 3^2 - 2 = 7$. Hence, our answer is $2^7 = 128$.

6. **[11]** Let A, E, H, L, T, and V be chosen independently and at random from the set $\left\{0, \frac{1}{2}, 1\right\}$. Compute the probability that $\lfloor T \cdot H \cdot E \rfloor = L \cdot A \cdot V \cdot A$.

Proposed by: Luke Robitaille

Answer: $\boxed{\frac{55}{81}}$

Solution: There are $3^3 - 2^3 = 19$ ways to choose L, A, and V such that $L \cdot A \cdot V \cdot A = 0$, since at least one of $\{L, A, V\}$ must be 0, and $3^3 - 1 = 26$ ways to choose T, H, and E such that $\lfloor T \cdot H \cdot E \rfloor = 0$, since at least one of $\{T, H, E\}$ must not be 1, for a total of $19 \cdot 26 = 494$ ways. There is only one way to make $\lfloor T \cdot H \cdot E \rfloor = L \cdot A \cdot V \cdot A = 1$, namely setting every variable equal to 1, so there are 495 total ways that work out of a possible $3^6 = 729$, for a probability of $\frac{55}{81}$.

7. **[11]** Let Ω be a sphere of radius 4 and Γ be a sphere of radius 2. Suppose that the center of Γ lies on the surface of Ω. The intersection of the surfaces of Ω and Γ is a circle. Compute this circle's circumference.

Proposed by: Luke Robitaille

Answer: $\boxed{\pi\sqrt{15}}$

Solution: Take a cross-section of a plane through the centers of Ω and Γ, call them O_1 and O_2, respectively. The resulting figure is two circles, one of radius 4 and center O_1, and the other with radius 2 and center O_2 on the circle of radius 4. Let these two circles intersect at A and B. Note that $\overline{AB}$ is a diameter of the desired circle, so we will find AB.

Focus on triangle O_1O_2A. The sides of this triangle are $O_1O_2 = O_1A = 4$ and $O_2A = 2$. The height from O_1 to AO_2 is $\sqrt{4^2 - 1^2} = \sqrt{15}$, and because $O_1O_2 = 2 \cdot AO_2$, the height from A to O_1O_2 is $\frac{\sqrt{15}}{2}$. Then the distance AB is two times this, or $\sqrt{15}$.

Thus, the circumference of the desired circle is $\pi\sqrt{15}$.

8. **[11]** Suppose a, b, and c are distinct positive integers such that $\sqrt{a\sqrt{b\sqrt{c}}}$ is an integer. Compute the least possible value of $a + b + c$.

Proposed by: Sean Li

Answer: $\boxed{7}$

Solution: First, check that no permutation of $(1, 2, 3)$ works, so the sum must be more than 6. Then since $(a, b, c) = (2, 4, 1)$ has $\sqrt{2\sqrt{4\sqrt{1}}} = 2$, the answer must be $2 + 4 + 1 = 7$.

9. **[13]** One hundred points labeled 1 to 100 are arranged in a 10×10 grid such that adjacent points are one unit apart. The labels are increasing left to right, top to bottom (so the first row has labels 1 to 10, the second row has labels 11 to 20, and so on).

   Convex polygon $\mathcal{P}$ has the property that every point with a label divisible by 7 is either on the boundary or in the interior of $\mathcal{P}$. Compute the smallest possible area of $\mathcal{P}$.

   Proposed by: Eric Shen

   Answer: $\boxed{63}$

   Solution: The vertices of the smallest $\mathcal{P}$ are located at the points on the grid corresponding to the numbers 7, 21, 91, 98, and 70. The entire grid has area 81, and the portion of the grid not in $\mathcal{P}$ is composed of three triangles of areas $6, 9, 3$. Thus the area of $\mathcal{P}$ is $81 - 6 - 9 - 3 = 63$.

10. **[13]** The number

$$316990099009901 = \frac{32016000000000001}{101}$$

   is the product of two distinct prime numbers. Compute the smaller of these two primes.

   Proposed by: Rishabh Das

   Answer: $\boxed{4002001}$

   Solution: Let $x = 2000$, so the numerator is

$$x^5 + x^4 + 1 = (x^2 + x + 1)(x^3 - x + 1).$$

   (This latter factorization can be noted by the fact that plugging in ω or ω^2 into $x^5 + x^4 + 1$ gives 0.) Then $x^2 + x + 1 = 4002001$ divides the numerator. However, it can easily by checked that 101 doesn't divide 4002001 (since, for example, $101 \nmid 1 - 20 + 0 - 4$), so 4002001 is one of the primes. Then the other one is

$$\frac{2000^3 - 2000 + 1}{101} \approx \frac{2000^3}{101} > 2000^2 \approx 4002001,$$

   so 4002001 is the smaller of the primes.

11. **[13]** The Fibonacci numbers are defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. Given 15 wooden blocks of weights $F_2, F_3, \ldots, F_{16}$, compute the number of ways to paint each block either red or blue such that the total weight of the red blocks equals the total weight of the blue blocks.

   Proposed by: Albert Wang

   Answer: $\boxed{32}$

   Solution: Partition the blocks into sets

$$\{F_2, F_3, F_4\}, \{F_5, F_6, F_7\}, \ldots, \{F_{14}, F_{15}, F_{16}\}.$$

   We can show by bounding that F_{16} belongs on the opposite side as F_{15} and F_{14}, and, in general, that F_{3k+1} is on the opposite side as F_{3k} and F_{3k-1}. Hence, it suffices to choose which side each of $F_4, F_7, \ldots, F_{16}$ go. This gives $2^5 = 32$ ways.

12. **[13]** The number 770 is written on a blackboard. Melody repeatedly performs *moves*, where a move consists of subtracting either 40 or 41 from the number on the board. She performs moves until the number is not positive, and then she stops. Let N be the number of sequences of moves that Melody could perform. Suppose $N = a \cdot 2^b$ where a is an odd positive integer and b is a nonnegative integer. Compute $100a + b$.

Proposed by: Albert Wang

Answer: $\boxed{318}$

Solution: Notice that if we use the 41 move nine times or less, we will have to make a total of $\lceil \frac{770}{40} \rceil = 20$ moves, and if we use it ten times or more, we will have to make a total of $\lfloor \frac{770}{40} \rfloor = 19$ moves. So, doing casework on the number of 40s we use gives

$$\underbrace{\binom{19}{0} + \binom{19}{1} + \binom{19}{2} + \cdots + \binom{19}{9}}_{19 \text{ moves}} + \underbrace{\frac{\binom{20}{10}}{2} + \binom{20}{11} + \binom{20}{11} + \cdots + \binom{20}{20}}_{20 \text{ moves}}.$$

Using the row sums of Pascal's triangle we have this sum equal to $\frac{2^{19}}{2} + \frac{2^{20}}{2} = 3 \cdot 2^{18}$. The answer is 318.

13. **[14]** Suppose a, b, c, and d are pairwise distinct positive perfect squares such that $a^b = c^d$. Compute the smallest possible value of $a + b + c + d$.

Proposed by: Luke Robitaille

Answer: $\boxed{305}$

Solution: Note that if a and c are divisible by more than one distinct prime, then we can just take the prime powers of a specific prime. Thus, assume a and c are powers of a prime p. Assume $a = 4^x$ and $c = 4^y$. Then $xb = yd$.

Because b and d are squares, the ratio of x to y is a square, so assume $x = 1$ and $y = 4$. We can't take $b = 4$ and $c = 1$, but we instead can take $b = 36$ and $c = 9$. It can be checked that other values of x and y are too big. This gives $4^{36} = 256^9$, which gives a sum of 305.

If a and c are powers of 9, then $\max(a, c) \geq 9^4$, which is already too big. Thus, 305 is optimal.

14. **[14]** Acute triangle ABC has circumcenter O. The bisector of $\angle ABC$ and the altitude from C to side AB intersect at X. Suppose that there is a circle passing through B, O, X, and C. If $\angle BAC = n^\circ$, where n is a positive integer, compute the largest possible value of n.

Proposed by: Luke Robitaille

Answer: $\boxed{67}$

Solution: We have $\angle XBC = B/2$ and $\angle XCB = 90^\circ - B$. Thus, $\angle BXC = 90^\circ + B/2$. We have $\angle BOC = 2A$, so

$$90^\circ + B/2 = 2A.$$

This gives $B = 4A - 180^\circ$, which gives $C = 360^\circ - 5A$.

In order for $0^\circ < B < 90^\circ$, we need $45^\circ < A < 67.5^\circ$. In order for $0^\circ < C < 90^\circ$, we require $54^\circ < A < 72^\circ$. The largest integer value in degrees satisfying these inequalities is $A = 67^\circ$.

15. **[14]** Let A and B be points in space for which $AB = 1$. Let $\mathcal{R}$ be the region of points P for which $AP \leq 1$ and $BP \leq 1$. Compute the largest possible side length of a cube contained within $\mathcal{R}$.

Proposed by: Henry Stennes

Answer: $\boxed{\frac{\sqrt{10}-1}{3}}$

Solution: Let h be the distance between the center of one sphere and the center of the opposite face of the cube. Let x be the side length of the cube. Then we can draw a right triangle by connecting the center of the sphere, the center of the opposite face of the cube, and one of the vertices that make up that face. This gives us $h^2 + (\frac{\sqrt{2}x}{2})^2 = 1$. Because the centers of the spheres are 1 unit apart,

$h = \frac{1}{2}x + \frac{1}{2}$, giving us the quadratic $(\frac{1}{2}x + \frac{1}{2})^2 + (\frac{\sqrt{2}x}{2})^2 = 1$. Solving yields $x = \frac{\sqrt{10}-1}{3}$.

16. **[14]** The graph of the equation $x + y = \lfloor x^2 + y^2 \rfloor$ consists of several line segments. Compute the sum of their lengths.

Proposed by: Sean Li

Answer: $\boxed{4 + \sqrt{6} - \sqrt{2}}$

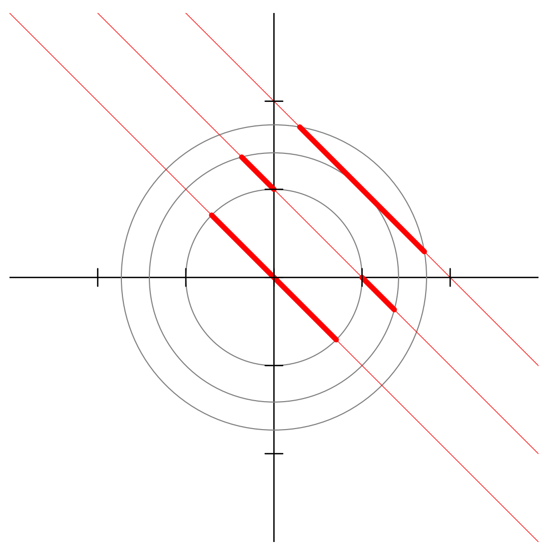
Solution: We split into cases on the integer $k = \lfloor x^2 + y^2 \rfloor$. Note that $x + y = k$ but $x^2 + y^2 \geq \frac{1}{2}(x+y)^2 = \frac{1}{2}k^2$ and $x^2 + y^2 < k + 1$, which forces $k \leq 2$.

If $k = 0$, the region defined by $0 \leq x^2 + y^2 < 1$ and $x + y = 0$ is the diameter from $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ to $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, which has length 2.

If $k = 1$, the region $1 \leq x^2 + y^2 < 2$ and $x + y = 1$ consists of two segments, which is the chord on $x^2 + y^2 = 2$ minus the chord on $x^2 + y^2 = 1$. The former has length $2\sqrt{(\sqrt{2})^2 - (\frac{\sqrt{2}}{2})^2} = \sqrt{6}$, and the latter has length $2\sqrt{1^2 - (\frac{\sqrt{2}}{2})^2} = \sqrt{2}$. So the total length here is $\sqrt{6} - \sqrt{2}$.

If $k = 2$, the region $2 \leq x^2 + y^2 < 3$ and $x + y = 1$ is the chord on $x^2 + y^2 = 3$, which has length $2\sqrt{(\sqrt{3})^2 - (\sqrt{2})^2} = 2$.

Our final answer is $2 + (\sqrt{6} - \sqrt{2}) + 2 = 4 + \sqrt{6} - \sqrt{2}$.



17. **[16]** An equilateral triangle lies in the Cartesian plane such that the x-coordinates of its vertices are pairwise distinct and all satisfy the equation $x^3 - 9x^2 + 10x + 5 = 0$. Compute the side length of the triangle.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\sqrt{68} = 2\sqrt{17}}$

Solution: Let three points be A, B, and C with x-coordinates a, b, and c, respectively. Let the circumcircle of $\triangle ABC$ meet the line $y = b$ at point P. Then, we have $\angle BPC = 60^\circ \implies PC =$

$\frac{2}{\sqrt{3}}(c-b)$. Similarly, $AP = \frac{2}{\sqrt{3}}(b-a)$. Thus, by the Law of Cosines,

$$\begin{aligned} AC^2 &= AP^2 + PC^2 - 2 \cdot AP \cdot PC \cos 120^\circ \\ &= \frac{4}{3}\big((c-b)^2 + (b-a)^2 + (c-b)(b-a)\big) \\ &= \frac{4}{3}\big(a^2 + b^2 + c^2 - ab - bc - ca\big) \\ &= \frac{4}{3}\big((a+b+c)^2 - 3(ab+bc+ca)\big). \end{aligned}$$

By Vieta's we have $a+b+c = 9$ and $ab+bc+ca = 10$, so we have $AC^2 = \frac{4}{3}(81-30) = 68$, implying that the answer is $\sqrt{68} = 2\sqrt{17}$.

18. **[16]** Elisenda has a piece of paper in the shape of a triangle with vertices A, B, and C such that $AB = 42$. She chooses a point D on segment AC, and she folds the paper along line BD so that A lands at a point E on segment BC. Then, she folds the paper along line DE. When she does this, B lands at the midpoint of segment DC. Compute the perimeter of the original unfolded triangle.

Proposed by: Drake Du, Eric Shen

Answer: $\boxed{168 + 48\sqrt{7}}$

Solution: Let F be the midpoint of segment DC.

Evidently $\angle ADB = 60^\circ = \angle BDE = \angle EDC$. Moreover, we have $BD = DF = FC$, $AD = DE$, and $AB = BE$. Hence angle bisector on BDC gives us that $BE = 42$, $EC = 84$, and hence angle bisector on ABC gives us that if $AD = x$ then $CD = 3x$. Now this gives $BD = 3x/2$, so thus the Law of Cosines on ADB gives $x = 12\sqrt{7}$.

Hence, $BC = 42 + 84 = 126$ and $AC = 4x = 48\sqrt{7}$. The answer is $42 + 126 + 48\sqrt{7} = 168 + 48\sqrt{7}$.

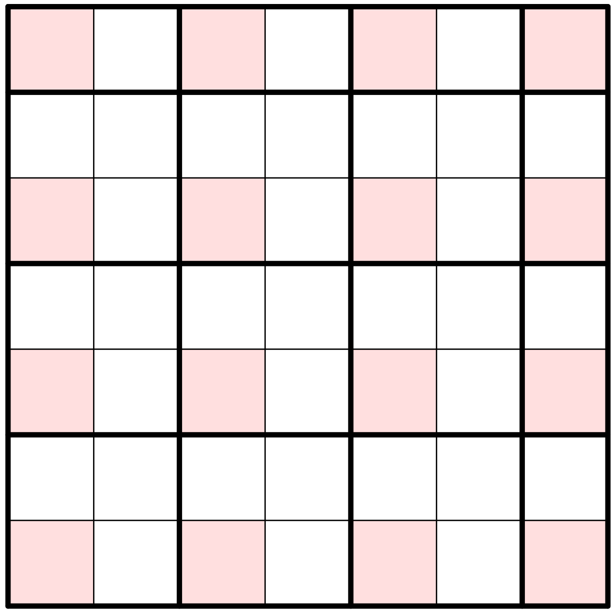
19. **[16]** Compute the number of ways to select 99 cells of a 19×19 square grid such that no two selected cells share an edge or vertex.

Proposed by: Sean Li

Answer: $\boxed{1000}$

Solution: We claim the number of ways to select $n^2 - 1$ such cells from a $(2n-1) \times (2n-1)$ grid is exactly n^3, which implies the answer to this question is 1000.

Partition the board into n^2 regions, as pictured. Also, shade red every cell in an odd row and column red, so there are n^2 red cells. Say a region is *blank* if it has no selected cell; *normal* if the selected cell is red; *up-wack* if the selected cell is above the red cell; and *right-wack* if the selected cell is to the right of the red cell. Note a 2×2 region could be both up-wack and right-wack.



The key idea is that we have at most one blank region, which restricts things significantly. We have two cases:

- **Case 1: no wack regions.** Then we pick a region to be blank, of which we have n^2 choices.
- **Case 2: some wack region.** Note that (1) any region directly above an up-wack region must be either blank or up-wack; and (2) any region directly to the right of a right-wack region must be either blank or right-wack. In particular, there is at most one wack region (and we cannot have any up-wack and right-wack regions), since every wack region corresponds to at least one blank region.

  Suppose some region is up-wack. There are n columns that could contain this up-wack region, and $\binom{n+1}{2}$ ways to pick an up-wack region and, optionally, a blank region above it. Similarly, there are $n\binom{n+1}{2}$ cases if there is some up-wack region, for a total of $2n\binom{n+1}{2}$ choices.

In total, we have $n^2 + 2n\binom{n+1}{2} = n^3$ possibilities, as desired.

20. **[16]** Five people take a true-or-false test with five questions. Each person randomly guesses on every question. Given that, for each question, a majority of test-takers answered it correctly, let p be the probability that every person answers exactly three questions correctly. Suppose that $p = \frac{a}{2^b}$ where a is an odd positive integer and b is a nonnegative integer. Compute $100a + b$.

    Proposed by: Evan Erickson, Leo Yao, Reagan Choi

    Answer: $\boxed{25517}$

    Solution: There are a total of 16^5 ways for the people to collectively ace the test. Consider groups of people who share the same problems that they got incorrect. We either have a group of 2 and a group of 3, or a group 5.

    In the first case, we can pick the group of two in $\binom{5}{2}$ ways, the problems they got wrong in $\binom{5}{2}$ ways. Then there are 3! ways for the problems of group 3. There are 600 cases here.

    In the second case, we can $5! \cdot 4!/2 = 120 \cdot 12$ ways to organize the five cycle ($4!/2$ to pick a cycle and 5! ways to assign a problem to each edge in the cycle).

    Thus, the solution is $\frac{255}{2^{17}}$ and the answer is 25517.

21. **[18]** Let x, y, and N be real numbers, with y nonzero, such that the sets $\{(x+y)^2, (x-y)^2, xy, x/y\}$ and $\{4, 12.8, 28.8, N\}$ are equal. Compute the sum of the possible values of N.

Proposed by: Sean Li

Answer: $\boxed{\frac{426}{5}}$

Solution: First, suppose that x and y were of different signs. Then $xy < 0$ and $x/y < 0$, but the set has at most one negative value, a contradiction. Hence, x and y have the same sign; without loss of generality, we say x and y are both positive.

Let $(s, d) := (x+y, x-y)$. Then the set given is equal to $\{s^2, d^2, \frac{1}{4}(s^2-d^2), \frac{s+d}{s-d}\}$. We split into two cases:

- **Case 1:** $\frac{s+d}{s-d} = N$. This forces $s^2 = 28.8$ and $d^2 = 12.8$, since $\frac{1}{4}(28.8 - 12.8) = 4$. Then $s = 12\sqrt{0.2}$ and $d = \pm 8\sqrt{0.2}$, so N is either $\frac{12+8}{12-8} = 5$ or $\frac{12-8}{12+8} = 0.2$.
- **Case 2:** $\frac{s+d}{s-d} \neq N$. Suppose $\frac{s+d}{s-d} = k$, so $(s, d) = ((k+1)t, (k-1)t)$ for some t. Then $s^2 : d^2 : \frac{1}{4}(s^2 - d^2) = (k+1)^2 : (k-1)^2 : k$. Trying $k = 4, 12.8, 28.8$ reveals that only $k = 4$ is possible, since $28.8 : 12.8 = (4-1)^2 : 4$. This forces $N = s^2 = \frac{5^2}{4} \cdot 12.8 = 80$.

Hence, our final total is $5 + 0.2 + 80 = 85.2$.

22. **[18]** Let a_0, a_1, a_2, $\ldots$ be an infinite sequence where each term is independently and uniformly random in the set $\{1, 2, 3, 4\}$. Define an infinite sequence b_0, b_1, b_2, $\ldots$ recursively by $b_0 = 1$ and $b_{i+1} = a_i^{b_i}$. Compute the expected value of the smallest positive integer k such that $b_k \equiv 1 \pmod 5$.

Proposed by: Raymond Feng

Answer: $\boxed{\frac{35}{16}}$

Solution: Do casework on what a_0 is.

If $a_0 = 1$ then $k = 1$.

If $a_0 = 4$ then $k = 2$.

If $a_0 = 3$ then

- if $a_1 = 1$, then $k = 2$
- if $a_1 = 2$ or 4, then $k = 3$
- if $a_1 = 3$, then you make no progress.

so in expectation it requires $E = (2 + 3 + (E+1) + 3)/4 \implies E = 3$.

If $a_0 = 2$ then

- if $a_1 = 1$ or 4, then $k = 2$
- if $a_1 = 2$, then $k = 3$
- if $a_1 = 3$, then it can be checked that if $a_2 = 1$ we get $k = 3$, if $a_2 = 2$ or 4 then $k = 4$, and if $a_2 = 3$ then we make no progress. Thus, this case is equivalent to the case of $a_0 = 3$ except shifted over by one, so it is $3 + 1 = 4$ in expectation.

So this case is $(2 + 3 + 4 + 2)/4$ in expectation.

This means the answer is $(1 + (11/4) + 3 + 2)/4 = 35/16$.

23. **[18]** A subset S of the set $\{1, 2, \ldots, 10\}$ is chosen randomly, with all possible subsets being equally likely. Compute the expected number of positive integers which divide the product of the elements of S. (By convention, the product of the elements of the empty set is 1.)

Proposed by: Sean Li

Answer: $\boxed{375/8}$

Solution: For primes $p = 2, 3, 5, 7$, let the random variable X_p denote the number of factors of p in the product of the elements of S, plus 1. Then we wish to find $\mathbb{E}(X_2X_3X_5X_7)$.

If there were only prime powers between 1 and 10, then all X_p would be independent. However, 6 and 10 are non-prime powers, so we will do casework on whether these elements are included:

- **Case 1: none included.** Note that $\mathbb{E}(X_2 \mid 6, 10 \notin S) = 1 + \frac{1}{2}(1 + 2 + 3) = 4$, since each of $\{2, 4, 8\}$ has a $1/2$ chance of being included in S. Similarly, $\mathbb{E}(X_3 \mid 6, 10 \notin S) = \frac{5}{2}$ and $\mathbb{E}(X_5 \mid 6, 10 \notin S) = \mathbb{E}(X_7 \mid 6, 10 \notin S) = \frac{3}{2}$. The values of X_2, X_3, X_5, and X_7 are independent given that $6, 10 \notin S$, so $\mathbb{E}(X_2X_3X_5X_7 \mid 6, 10 \notin S) = 4 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} = \frac{45}{2}$.
- **Case 2: 6 included.** Now, we have $\mathbb{E}(X_2 \mid 6 \in S, 10 \notin S) = 5$ and $\mathbb{E}(X_3 \mid 6 \in S, 10 \notin S) = \frac{7}{2}$, since we know $6 \in S$. We still have $\mathbb{E}(X_5 \mid 6 \in S, 10 \notin S) = \mathbb{E}(X_7 \mid 6 \in S, 10 \notin S) = \frac{3}{2}$. The values of X_2, X_3, X_5, and X_7 are independent given that $6 \in S$ but $10 \notin S$, so $\mathbb{E}(X_2X_3X_5X_7 \mid 6 \in S, 10 \notin S) = 5 \cdot \frac{7}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} = \frac{315}{8}$.
- **Case 3: 10 included.** We have $\mathbb{E}(X_2 \mid 10 \in S, 6 \notin S) = 5$ and $\mathbb{E}(X_5 \mid 10 \in S, 6 \notin S) = \frac{5}{2}$, since we know $10 \in S$. We also have $\mathbb{E}(X_3 \mid 10 \in S, 6 \notin S) = \frac{5}{2}$ and $\mathbb{E}(X_7 \mid 10 \in S, 6 \notin S) = \frac{3}{2}$, hence $\mathbb{E}(X_2X_3X_5X_7 \mid 10 \in S, 6 \notin S) = 5 \cdot \frac{5}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} = \frac{375}{8}$.
- **Case 4: 6 and 10 included.** We have $\mathbb{E}(X_2 \mid 6, 10 \in S) = 6$, $\mathbb{E}(X_3 \mid 6, 10 \in S) = \frac{7}{2}$, and $\mathbb{E}(X_5 \mid 6, 10 \in S) = \frac{5}{2}$. We still have $\mathbb{E}(X_7 \mid 6, 10 \in S) = \frac{3}{2}$, hence $\mathbb{E}(X_2X_3X_5X_7 \mid 6, 10 \in S) = 6 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} = \frac{315}{4}$.

The average of these quantities is $\frac{1}{4}(\frac{45}{2} + \frac{315}{8} + \frac{375}{8} + \frac{315}{4}) = \frac{375}{8}$, as desired.

24. **[18]** Let $AXBY$ be a cyclic quadrilateral, and let line AB and line XY intersect at C. Suppose $AX \cdot AY = 6$, $BX \cdot BY = 5$, and $CX \cdot CY = 4$. Compute AB^2.

Proposed by: Luke Robitaille

Answer: $\boxed{\frac{242}{15}}$

Solution: Observe that

$$\triangle ACX \sim \triangle YCB \implies \frac{AC}{AX} = \frac{CY}{BY}$$
$$\triangle ACY \sim \triangle XCB \implies \frac{AC}{AY} = \frac{CX}{BX}$$

Mulitplying these two equations together, we get that

$$AC^2 = \frac{(CX \cdot CY)(AX \cdot AY)}{BX \cdot BY} = \frac{24}{5}.$$

Analogously, we obtain that

$$BC^2 = \frac{(CX \cdot CY)(BX \cdot BY)}{AX \cdot AY} = \frac{10}{3}.$$

Hence, we have

$$AB = AC + BC = \sqrt{\frac{24}{5}} + \sqrt{\frac{10}{3}} = \frac{11\sqrt{30}}{15},$$

implying the answer.

25. **[20]** The *spikiness* of a sequence $a_1, a_2, \ldots, a_n$ of at least two real numbers is the sum $\sum_{i=1}^{n-1} |a_{i+1} - a_i|$. Suppose $x_1, x_2, \ldots, x_9$ are chosen uniformly and randomly from the interval $[0, 1]$. Let M be the largest possible value of the spikiness of a permutation of $x_1, x_2, \ldots, x_9$. Compute the expected value of M.

Proposed by: Gabriel Wu

Answer: $\boxed{\frac{79}{20}}$

Solution: Our job is to arrange the nine numbers in a way that maximizes the spikiness. Let an element be a peak if it is higher than its neighbor(s) and a valley if it is lower than its neighbor(s). It is not hard to show that an optimal arrangement has every element either a peak or a valley (if you have some number that is neither, just move it to the end to increase spikiness). Since 9 is odd, there are two possibilities: the end points are either both peaks or both valleys.

Sort the numbers from least to greatest: $x_1, \ldots, x_9$. If we arrange them in such a way that it starts and ends with peaks, the factor of x_i added to the final result will be $[-2, -2, -2, -2, 1, 1, 2, 2, 2]$, respectively. If we choose the other way (starting and ending with valleys), we get $[-2, -2, -2, -1, -1, 2, 2, 2, 2]$. Notice that both cases have a base value of $[-2, -2, -2, -1, 0, 1, 2, 2, 2]$, but then we add on $\max(x_6 - x_5, x_5 - x_4)$. Since the expected value of x_i is $\frac{i}{10}$, our answer is $-\frac{2}{10}(1+2+3) - \frac{4}{10} + \frac{6}{10} + \frac{2}{10}(7+8+9) + \mathbb{E}(\max(x_6 - x_5, x_5 - x_4))$. This last term actually has value $\frac{3}{4}\mathbb{E}(x_6 - x_4) = \frac{3}{4} \cdot \frac{2}{10}$. This is because if we fix all values except x_5, then x_5 is uniformly distributed in $[x_4, x_6]$. Geometric probability tells us that the distance from x_5 to its farthest neighbor is $\frac{3}{4}$ to total distance betwen its two neighbors $(x_6 - x_4)$. We add this all up to get $\frac{79}{20}$.

26. **[20]** Let $PABC$ be a tetrahedron such that $\angle APB = \angle APC = \angle BPC = 90°$, $\angle ABC = 30°$, and AP^2 equals the area of triangle ABC. Compute $\tan \angle ACB$.

Proposed by: Luke Robitaille

Answer: $\boxed{8 + 5\sqrt{3}}$

Solution: Observe that

$$\begin{aligned}\frac{1}{2} \cdot AB \cdot AC \cdot \sin \angle BAC = [ABC] &= AP^2 \\ &= \frac{1}{2}(AB^2 + AC^2 - BC^2) \\ &= AB \cdot AC \cdot \cos \angle BAC,\end{aligned}$$

so $\tan \angle BAC = 2$. Also, we have $\tan \angle ABC = \frac{1}{\sqrt{3}}$. Also, for any angles α, β, γ summing to $180°$, one can see that $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \cdot \tan \beta \cdot \tan \gamma$. Thus we have $\tan \angle ACB + 2 + \frac{1}{\sqrt{3}} = \tan \angle ACB \cdot 2 \cdot \frac{1}{\sqrt{3}}$, so $\tan \angle ACB = 8 + 5\sqrt{3}$.

27. **[20]** Suppose $m > n > 1$ are positive integers such that there exist n complex numbers $x_1, x_2, \ldots, x_n$ for which

- $x_1^k + x_2^k + \cdots + x_n^k = 1$ for $k = 1, 2, \ldots, n-1$;
- $x_1^n + x_2^n + \cdots + x_n^n = 2$; and
- $x_1^m + x_2^m + \cdots + x_n^m = 4$.

Compute the smallest possible value of $m + n$.

Proposed by: Rishabh Das

Answer: $\boxed{34}$

Solution: Let $S_k = \sum_{j=1}^{n} x_j^k$, so $S_1 = S_2 = \cdots = S_{n-1} = 1$, $S_n = 2$, and $S_m = 4$. The first of these conditions gives that $x_1, \ldots, x_n$ are the roots of $P(x) = x^n - x^{n-1} - c$ for some constant c. Then $x_i^n = x_i^{n-1} + c$, and thus

$$2 = S_n = S_{n-1} + cn = 1 + cn,$$

so $c = \frac{1}{n}$.

Thus, we have the recurrence $S_k = S_{k-1} + \frac{S_{k-n}}{n}$. This gives $S_{n+j} = 2 + \frac{j}{n}$ for $0 \le j \le n-1$, and then $S_{2n} = 3 + \frac{1}{n}$. Then $S_{2n+j} = 3 + \frac{2j+1}{n} + \frac{j^2+j}{2n^2}$ for $0 \le j \le n-1$. In particular, $S_{3n-1} > 4$, so we have $m \in [2n, 3n-1]$. Let $m = 2n + j$. Then

$$3 + \frac{2j+1}{n} + \frac{j^2+j}{2n^2} = 4 \implies 2n^2 - 2n(2j+1) - (j^2+j) = 0.$$

Viewing this as a quadratic in n, the discriminant $4(2j+1)^2 + 8(j^2+j) = 24j^2 + 24j + 4 = 4(6j^2+6j+1)$ must be a perfect square, so $6j^2 + 6j + 1$ is a square. Then

$$6j^2 + 6j + 1 = y^2 \implies 12j^2 + 12j + 2 = 2y^2 \implies 3(2j+1)^2 - 2y^2 = 1.$$

The case $j = 0$ gives $n = 1$, a contradiction. After this, the smallest j that works is $j = 4$ (and $y = 11$).

Plugging this back into our quadratic,

$$2n^2 - 18n - 20 = 0 \implies n^2 - 9n - 10 = 0,$$

so $n = 10$. Then $m = 2n + j = 24$, so $m + n = 34$.

28. **[20]** Suppose $ABCD$ is a convex quadrilateral with $\angle ABD = 105^\circ$, $\angle ADB = 15^\circ$, $AC = 7$, and $BC = CD = 5$. Compute the sum of all possible values of BD.

Proposed by: Luke Robitaille

Answer: $\boxed{\sqrt{291}}$

Solution: Let O be the cirumcenter of triangle ABD. By the inscribed angle theorem, $\angle AOC = 90^\circ$ and $\angle BOC = 60^\circ$. Let $AO = BO = CO = x$ and $CO = y$. By the Pythagorean theorem on triangle AOC,

$$x^2 + y^2 = 49$$

and by the Law of Cosines on triangle BOC,

$$x^2 - xy + y^2 = 25.$$

It suffices to find the sum of all possible values of $BD = \sqrt{3}x$.

Since the two conditions on x and y are both symmetric, the answer is equal to

$$\sqrt{3}(x+y) = \sqrt{9(x^2+y^2) - 6(x^2 - xy + y^2)} = \boxed{\sqrt{291}}.$$

It is easy to check that both solutions generate valid configurations.

29. **[23]** Let $P_1(x)$, $P_2(x)$, $\ldots$, $P_k(x)$ be monic polynomials of degree 13 with integer coefficients. Suppose there are pairwise distinct positive integers n_1, n_2, $\ldots$, n_k for which, for all positive integers i and j less than or equal to k, the statement "n_i divides $P_j(m)$ for every integer m" holds if and only if $i = j$. Compute the largest possible value of k.

Proposed by: Reagan Choi

Answer: $\boxed{144}$

Solution: We first consider which integers can divide a polynomial $P_i(x)$ for all x. Assume that $c|P_i(x)$ for all x. Then, c must also divide the finite difference $Q(x) = Q_i(x+1) - Q_i(x)$. Since $Q_i(x)$ is degree 13 and monic, the leading term of $Q(x)$ is the leading term of $(x+1)^{13} - x^{13}$, which is $13x^{12}$.

Continuing this process finding finite differences, we see that c must divide $R(x) = Q(x+1) - Q(x)$, which has a leading term $13 \cdot 12x^{11}$. At the end, we will see that $c|13!$, so these are the only possible values of c. To show that all of these values of c work, consider the polynomial $P_i(x) = x(x+1)\cdots(x+12)+c$. It can be easily seen that the product of thirteen consecutive integers is always divisible by 13!, so this polynomial is always divisible by c and nothing more, as $P_i(0) = c$.

Now, we find the maximum possible value of k. Note that if two polynomials have values of n_i and n_j, we cannot have $n_i|n_j$ since then $n_i|P_j(x)$ for all x. Hence, we wish to find as many values of $c|13!$ as possible that do not divide each other.

We prime factorize $13! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. We claim that the maximum number of polynomials is $k = 144$. This is a maximum since there are $6 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 144$ odd factors of 13!; and if two values n_i and n_j have the same odd component by the Pigeonhole Principle, then either $\frac{n_i}{n_j}$ or $\frac{n_j}{n_i}$ is a power of 2. In addition, $k = 144$ is attainable by taking $2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e \cdot 13^f$ for $a+b+c+d+e+f = 10$, in which there is exactly one solution for a for each of the 144 valid quintuples (b, c, d, e, f). Hence, $k = 144$ is the maximum.

30. **[23]** Five pairs of twins are randomly arranged around a circle. Then they perform zero or more *swaps*, where each swap switches the positions of two adjacent people. They want to reach a state where no one is adjacent to their twin. Compute the expected value of the smallest number of swaps needed to reach such a state.

Proposed by: Amy Feng

Answer: $\boxed{\frac{926}{945}}$

Solution: First, let's characterize the minimum number of swaps needed given a configuration. Each swap destroys 0, 1, or 2 adjacent pairs. If at least one pair is destroyed, no other adjacent pairs can be formed. Therefore, we only care about the count of adjacent pairs and should never create any new ones. In a maximal block of k adjacent pairs, defined as k consecutive (circular) adjacent pairs, we need at least $\lceil \frac{k}{2} \rceil$ swaps. Maximal blocks are independent as we never create new ones. Thus, we need $\sum_i \lceil \frac{k_i}{2} \rceil$ over maximal blocks.

Now we focus on counting the desired quantity over all configurations. As the expression above is linear and because expectation is linear, our answer is the sum of the number of 1−maximal blocks, 2−maximal blocks, ..., 5−maximal blocks. Note that there can't be a 4− maximal block. This can be computed as

$$\mathbb{E}[AA] - \mathbb{E}[AABB] + \mathbb{E}[AABBCC] - \mathbb{E}[AABBCCDDEE],$$

where $AA\ldots$ denotes a (not necessarily maximal) block of adjacent pairs and $\mathbb{E}[AA\ldots]$ is the expected count of such. (This counts a block of AA as 1, a block of $AABB$ as 1, a block of $AABBCC$ as 2, and a block of $AABBCCDDEE$ as 3 overall, as desired).

Lastly, we compute this quantity. Say there's n pairs. Let's treat each of the $2n$ people as distinguishable. The expected number of k consecutive adjacent pairs (not necessarily as a maximal block) equals

$$\frac{1}{2^n} n \binom{n}{k} k!(2n-2k)!2^k$$

The first n comes from choosing the start of this chain, $\binom{n}{k}$ from choosing which pairs are in this chain, $k!$ from permuting these pairs, 2^k from ordering the people in each pair in the chain, and $(2n-2k)!$ from permuting the other people.

We plug in $n=5$ to obtain $\frac{926}{945}$.

31. **[23]** Let

$$P = \prod_{i=0}^{2016} \left(i^3 - i - 1\right)^2.$$

The remainder when P is divided by the prime 2017 is not zero. Compute this remainder.

Proposed by: Srinath Mahankali

Answer: $\boxed{1994}$

Solution 1: Let $Q(x) = x^3 - x - 1 = (x-a)(x-b)(x-c)$, for $a, b, c \in F_{p^3}$. Then, we can write

$$P = \prod_{i=0}^{2016} (i-a)(i-b)(i-c).$$

If we consider each root separately, then

$$P = -(a^{2017} - a)(b^{2017} - b)(c^{2017} - c).$$

The key observation is that $a^{2017}, b^{2017}, c^{2017}$ is some nontrivial cycle of a, b, c. This is because by Frobenius' identity, $(a+b)^p = a^p + b^p$. So, if $P(x) = 0, P(x^{2017}) = 0$. But, $x^{2017} \neq x$, since $x \notin \mathbb{F}_p$. This implies the claim. So, in the end, we wish to compute

$$(b-a)^2(c-b)^2(a-c)^2.$$

Note that $P'(x) = 3x^2 - 1 = (x-b)(x-c) + (x-a)(x-b) + (x-c)(x-a)$. So it suffices to compute

$$(1-3a^2)(1-3b^2)(1-3c^2) = -27P\left(\frac{1}{\sqrt{3}}\right)P\left(-\frac{1}{\sqrt{3}}\right).$$

We can compute this is equal to $-27\left(1 - \frac{2}{3\sqrt{3}}\right)\left(1 + \frac{2}{3\sqrt{3}}\right) = -23$.

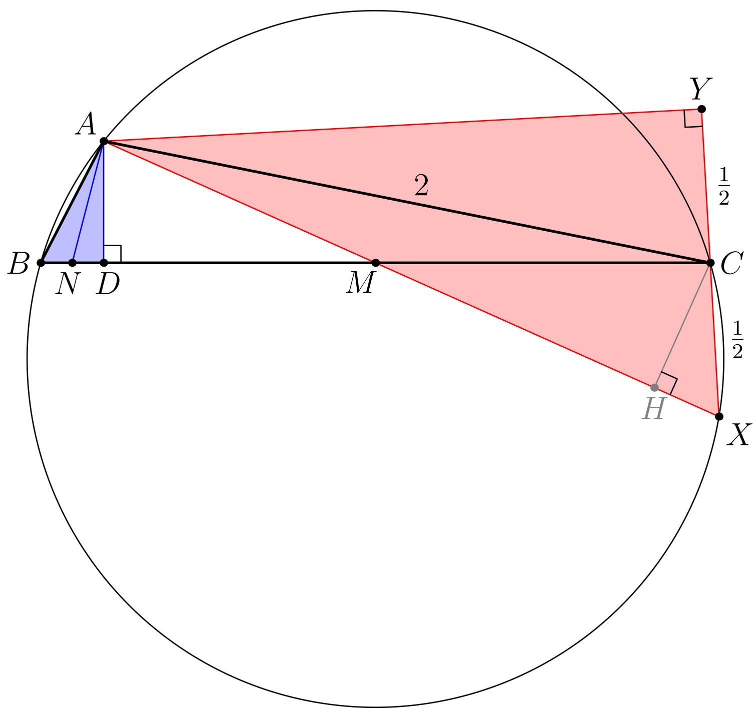
Alternatively, this is the discriminant of $P(x)$. We utilize the well-known formula that the discriminant of $x^3 + ax + b$ is $-4a^3 - 27b^2 = -23$. So, the answer is 1994.

32. **[23]** Let ABC be a triangle with $\angle BAC > 90°$. Let D be the foot of the perpendicular from A to side BC. Let M and N be the midpoints of segments BC and BD, respectively. Suppose that $AC = 2$, $\angle BAN = \angle MAC$, and $AB \cdot BC = AM$. Compute the distance from B to line AM.

Proposed by: Luke Robitaille, Pitchayut Saengrungkongka

Answer: $\boxed{\frac{\sqrt{285}}{38}}$

Solution:



Extend AM to meet the circumcircle of $\triangle ABC$ at X. Then, we have $\triangle ABM \sim \triangle CXM$, which implies that $\frac{CX}{CM} = \frac{AB}{AM}$. Using the condition $AB \cdot BC = AM$, we get that $CX = \frac{1}{2}$.

Now, the key observation is that $\triangle ANB \sim \triangle ACX$. Thus, if we let Y be the reflection of X across point C, we get that $\angle AYC = 90^\circ$. Thus, Pythagorean's theorem gives $AY = \sqrt{AC^2 - CY^2} = \sqrt{15/4}$ and $AX^2 = \sqrt{AY^2 + XY^2} = \sqrt{19/4}$.

Finally, note that the distance from B and C to line AM are equal. Thus, let H be the foot from C to AM. Then, from $\triangle XCH \sim \triangle XAY$, we get that

$$CH = AY \cdot \frac{CX}{AX} = \frac{\sqrt{15}}{2} \cdot \frac{1/2}{\sqrt{19}/2} = \frac{\sqrt{285}}{38}.$$

33. **[25]** Given a function f, let $\pi(f) = f \circ f \circ f \circ f \circ f$. The attached sheet has the graphs of ten smooth functions from the interval $(0, 1)$ to itself. The left-hand side consists of five functions:

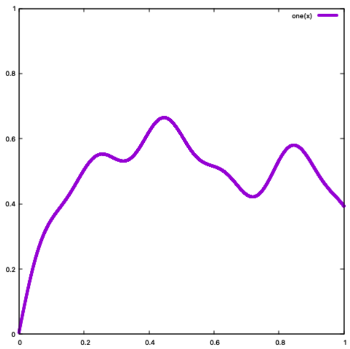
- $F_1(x) = 0.005 + \frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \frac{1}{8}\sin 8x + \frac{1}{16}\sin 16x + \frac{1}{32}\sin 32x$;
- $F_2(x) = F_1(F_1(x + 0.25))$;
- $F_3(x) = F_1((1-x)F_1((1-x)^2))$;
- $F_4(x) = F_1(x) + 0.05\sin(2\pi x)$;
- $F_5(x) = F_1(x + 1.45) + 0.65$.

The right-hand side consists of the five functions A, B, C, D, and E, which are $\pi(F_1), \ldots, \pi(F_5)$ in some order. Compute which of the functions $\{A, B, C, D, E\}$ correspond to $\pi(F_k)$ for $k = 1, 2, 3, 4, 5$.

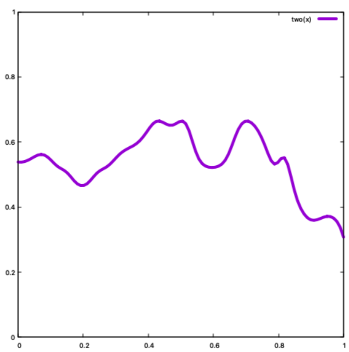
Your answer should be a five-character string containing A, B, C, D, E, or X for blank. For instance, if you think $\pi(F_1) = A$ and $\pi(F_5) = E$, then you would answer $AXXXE$. If you attempt to identify n functions and get them **all** correct, then you will receive n^2 points. Otherwise, you will receive 0 points.

Proposed by: Sean Li

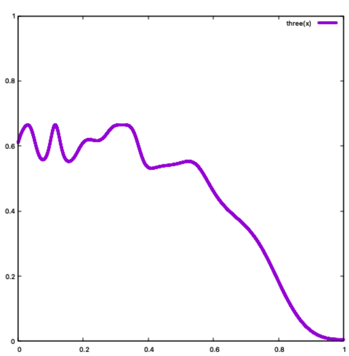
$F_1.$



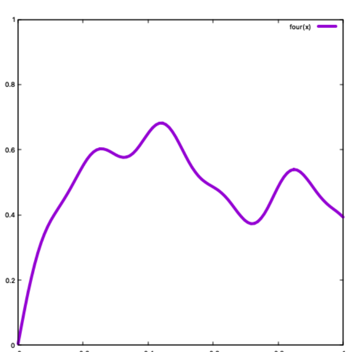
$F_2.$



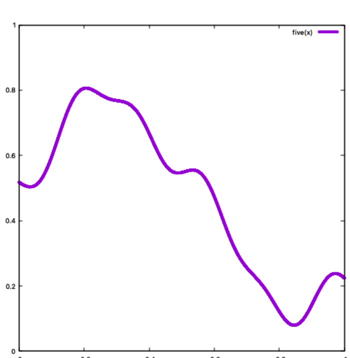
$F_3.$



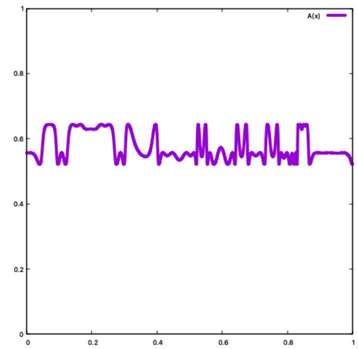
$F_4.$



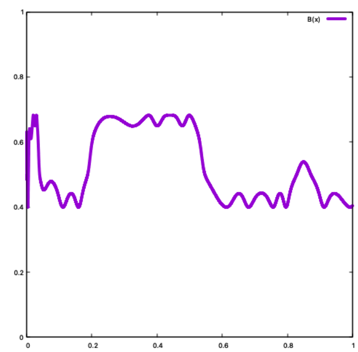
$F_5.$



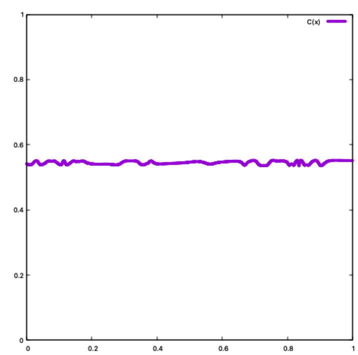
$A.$



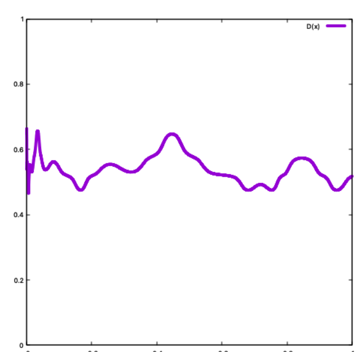
$B.$



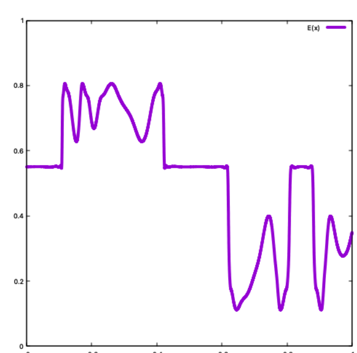
$C.$



$D.$



$E.$



Answer: $\boxed{DACBE}$

Solution: First, note that F_5 and E are the only functions whose range is outside the range $(0, 0.7)$, so they must be matched.

For graphs F_2 and F_3, split the square $[0,1]^2$ into a three by three grid of $\frac{1}{3} \times \frac{1}{3}$ squares. The only squares that the graphs pass through are $[\frac{1}{3}, \frac{2}{3}) \times I$ and possibly $[\frac{2}{3}, 1] \times [0, \frac{1}{3})$, so after five iterates the range of these graphs stabilizes to $f([\frac{1}{3}, \frac{2}{3}))$. Graph F_3 corresponds to the flatter graph C, while graph F_2 to the bumpier graph A. This is good enough for 9 points.

It remains to distinguish between F_1 and F_4. The details are a bit messy, so here are two ideas:

- Split the grid in a 5×5 grid of $\frac{1}{5}$-by-$\frac{1}{5}$ squares. It is clear these will distinguish F_1 and F_4 since B does not pass through $[0.2, 0.4) \times [0.4, 0.6)$ but D does not.
- Observing the definitions of F_1 and F_4, we have that the phases of the waves that make up F_1 "agree" much more than they do in F_4, since π is irrational. This suggests that $\pi(F_4)$ will have more chaotic behavior than $\pi(F_1)$.

The final answer is $DACBE$.

Remark. The graphs were made using `gnuplot`.

34. **[25]** The number 2027 is prime. For $i = 1, 2, \ldots, 2026$, let p_i be the smallest prime number such that $p_i \equiv i \pmod{2027}$. Estimate $\max(p_1, \ldots, p_{2026})$.

Submit a positive integer E. If the correct answer is A, you will receive $\lfloor 25 \min((E/A)^8, (A/E)^8) \rfloor$ points. (If you do not submit a positive integer, you will receive zero points for this question.)

Proposed by: Luke Robitaille, Brian Liu, Maxim Li, Misheel Otgonbayar, Sean Li, William Wang

Answer: $\boxed{113779}$

Solution: In this solution, all logs are in base e. Let $p_1, p_2, \ldots$ be the primes in sorted order. Let $q_i = p_i \mod 2027$. Since the residues of primes modulo 2027 should be uniformly distributed, we can make the probabilistic approximation that the q_i are random variables uniformly distributed among $1, \ldots, 2026$. This becomes the famous "coupon collector" problem: the random variables q_i are coupons with 2026 different types, and we keep collecting coupons until we have encountered one of each type. In other words, we seek to find the smallest k such that $\{q_1, \ldots, q_k\} = \{1, \ldots, 2026\}$, and then the answer to the problem is p_k.

It is known that the expected value of k is $2026(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2026}) \approx 2026 \log 2026$. This is because we must draw an expected $\frac{2026}{2026}$ coupons until we get our first distinct coupon type, then an expected $\frac{2026}{2025}$ coupons until we get our second new coupon type, and so on. The standard deviation of k is a small fraction of its its expectation, so we can safely assume that k is approximately $2026 \log 2026$. Since the n-th prime is approximately $n \log n$, our estimate is

$$\begin{aligned} E &\approx 2026 \log 2026 \log(2026 \log 2026) \\ &\approx 2026 \log^2 2026 \\ &\approx 117448 \end{aligned}$$

This achieves $A/E \approx 0.969$, which scores 19 out of 25 points.

35. **[25]** The Fibonacci numbers are defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. Given 30 wooden blocks of weights $\sqrt[3]{F_2}, \sqrt[3]{F_3}, \ldots, \sqrt[3]{F_{31}}$, estimate the number of ways to paint each block either red or blue such that the total weight of the red blocks and the total weight of the blue blocks differ by at most 1.

Submit a positive integer E. If the correct answer is A, you will receive $\lfloor 25\min((E/A)^8,(A/E)^8)\rfloor$ points. (If you do not submit a positive integer, you will receive zero points for this question.)

Proposed by: Albert Wang, Luke Robitaille

Answer: $\boxed{3892346}$

Solution: To get within an order of magnitude, one approach is to let X_n be a random variable which takes the value $\pm\sqrt[3]{F_n}$, with the sign chosen uniformly at random. We want the probability that $S=\sum_{i=2}^{31} X_i$ is in $[-1,1]$. We can attempt to approximate the distribution of S as normal (this is loosely justified because it is the sum of many independent random variables). Using the approximation $F_n \approx \frac{1}{\sqrt{5}}\varphi^n$ for $\phi=\frac{1+\sqrt{5}}{2}$, the variance of S is:

$$\begin{aligned}
\operatorname{Var}(S) &= \sum_{i=2}^{31} \operatorname{Var}(X_i)\\
&= \sum_{i=2}^{31} F_i^{2/3}\\
&\approx \sum_{i=2}^{31} 5^{-1/3}\varphi^{2i/3}\\
&\approx 5^{-1/3}\cdot\left(\frac{\varphi^{62/3}}{1-\varphi^{-2/3}}\right)
\end{aligned}$$

Now, we use the fact that if $\frac{1}{\sqrt{\operatorname{Var}(S)}}S$ is standard normal, then the probability that $S\in[-1,1]$ is approximately

$$\frac{1}{\sqrt{2\pi}}\cdot\frac{2}{\sqrt{\operatorname{Var}(S)}} \approx \sqrt{\frac{2\cdot 5^{1/3}}{\pi}}\cdot\frac{\sqrt{1-\varphi^{-2/3}}}{\varphi^{31/3}}$$

When we multiply this by 2^{30}, we get an approximation of $E\approx 4064598$, which achieves $A/E\approx 0.96$ and would score 17 out of 25 points.

36. **[25]** After the Guts round ends, the HMMT organizers will calculate A, the total number of points earned over all participating teams on questions 33, 34, and 35 of this round (that is, the other estimation questions). Estimate A.

Submit a positive integer E. You will receive $\max(0,25-3\cdot|E-A|)$ points. (If you do not submit a positive integer, you will receive zero points for this question.)

For your information, there are about 70 teams competing.

Proposed by: Luke Robitaille, Sean Li

Answer: $\boxed{13}$

Solution: Only 8 teams scored a positive number of combined points on questions 33, 34, and 35. A total of 3 points were scored on question 33, 6 points on question 34, and 4 points on question 35. Extended results can be found in our archive.