HMIC 2023
April 15–22, 2023

1. **[6]** Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Find, with proof, all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that, for all positive rational numbers x and y, we have

$$f(x) = f(x + y) + f(x + x^2 f(y)).$$

2. **[7]** A prime number p is *mundane* if there exist positive integers a and b less than $\frac{p}{2}$ such that $\frac{ab-1}{p}$ is a positive integer. Find, with proof, all prime numbers that are not mundane.

3. **[9]** Triangle ABC has incircle ω and A-excircle ω_A. Circle γ_B passes through B and is externally tangent to ω and ω_A. Circle γ_C passes through C and is externally tangent to ω and ω_A. If γ_B intersects line BC again at D, and γ_C intersects line BC again at E, prove that $BD = EC$.

4. **[9]** Let $n > 1$ be a positive integer. Claire writes n distinct positive real numbers $x_1, x_2, \ldots, x_n$ in a row on a blackboard. In a *move*, William can erase a number x and replace it with either $\frac{1}{x}$ or $x + 1$ at the same location. His goal is to perform a sequence of moves such that after he is done, the numbers are strictly increasing from left to right.

   (a) Prove that there exists a positive constant A, independent of n, such that William can always reach his goal in at most $An \log n$ moves.

   (b) Prove that there exists a positive constant B, independent of n, such that Claire can choose the initial numbers such that William cannot attain his goal in less than $Bn \log n$ moves.

5. **[11]** Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers such that, for all positive integers m and n, we have that a_{m+n} divides $a_m a_n - 1$. Prove that there exists an integer C such that, for all positive integers $k > C$, we have $a_k = 1$.

Time limit: 4 hours.