HMMT November 2023

November 11, 2023

General Round

1. Four people are playing rock-paper-scissors. They each play one of the three options (rock, paper, or scissors) independently at random, with equal probability of each choice. Compute the probability that someone beats everyone else.

   (In rock-paper-scissors, a player that plays rock beats a player that plays scissors, a player that plays paper beats a player that plays rock, and a player that plays scissors beats a player that plays paper.)

2. A regular n-gon $P_1P_2\ldots P_n$ satisfies $\angle P_1P_7P_8 = 178^\circ$. Compute n.

3. Compute the number of positive four-digit multiples of 11 whose sum of digits (in base ten) is divisible by 11.

4. Suppose that a and b are real numbers such that the line $y = ax + b$ intersects the graph of $y = x^2$ at two distinct points A and B. If the coordinates of the midpoint of AB are $(5, 101)$, compute $a + b$.

5. On an 8×8 chessboard, 6 black rooks and k white rooks are placed on different cells so that each rook only attacks rooks of the opposite color. Compute the maximum possible value of k.

   (Two rooks *attack* each other if they are in the same row or column and no rooks are between them.)

6. Let $ABCD$ be a square of side length 5. A circle passing through A is tangent to segment CD at T and meets AB and AD again at $X \neq A$ and $Y \neq A$, respectively. Given that $XY = 6$, compute AT.

7. Compute all ordered triples (x, y, z) of real numbers satisfying the following system of equations:

$$\begin{aligned} xy + z &= 40 \\ xz + y &= 51 \\ x + y + z &= 19. \end{aligned}$$

8. Mark writes the expression $\sqrt{d}$ for each positive divisor d of 8! on the board. Seeing that these expressions might not be worth points on HMMT, Rishabh simplifies each expression to the form $a\sqrt{b}$, where a and b are integers such that b is not divisible by the square of a prime number. (For example, $\sqrt{20}$, $\sqrt{16}$, and $\sqrt{6}$ simplify to $2\sqrt{5}$, $4\sqrt{1}$, and $1\sqrt{6}$, respectively.) Compute the sum of $a + b$ across all expressions that Rishabh writes.

9. An entry in a grid is called a *saddle point* if it is the largest number in its row and the smallest number in its column. Suppose that each cell in a 3×3 grid is filled with a real number, each chosen independently and uniformly at random from the interval $[0, 1]$. Compute the probability that this grid has at least one saddle point.

10. Let $ABCD$ be a convex trapezoid such that $\angle ABC = \angle BCD = 90^\circ$, $AB = 3$, $BC = 6$, and $CD = 12$. Among all points X inside the trapezoid satisfying $\angle XBC = \angle XDA$, compute the minimum possible value of CX.