HMMT November 2023

November 11, 2023

General Round

1. Four people are playing rock-paper-scissors. They each play one of the three options (rock, paper, or scissors) independently at random, with equal probability of each choice. Compute the probability that someone beats everyone else.

   (In rock-paper-scissors, a player that plays rock beats a player that plays scissors, a player that plays paper beats a player that plays rock, and a player that plays scissors beats a player that plays paper.)

   Proposed by: Evan Erickson

   Answer: $\boxed{\frac{4}{27}}$

   Solution: As the four players and three events are symmetric, the probability a particular player makes a particular move and beats everyone else is the same regardless of the choice of player or move. So, focusing on one such scenario, the desired probability is 12 times the probability that player 1 plays rock and beats everyone else.

   In this case, player 1 plays rock and all other players must play scissors. All four of these events have probability $\frac{1}{3}$, so this scenario has probability $\frac{1}{3^4} = \frac{1}{81}$. Thus,

$$\mathbb{P}(\text{one beats all}) = 12 \cdot \frac{1}{81} = \frac{4}{27}.$$

2. A regular n-gon $P_1P_2 \ldots P_n$ satisfies $\angle P_1P_7P_8 = 178°$. Compute n.

   Proposed by: Derek Liu

   Answer: $\boxed{630}$

   Solution: Let O be the center of the n-gon. Then

$$\angle P_1OP_8 = 2(180° - \angle P_1P_7P_8) = 4° = \frac{360°}{90},$$

   which means the arc $\widehat{P_1P_8}$ that spans 7 sides of the n-gon also spans $1/90$ of its circumcircle. Thus $n = 7 \cdot 90 = 630$.

3. Compute the number of positive four-digit multiples of 11 whose sum of digits (in base ten) is divisible by 11.

   Proposed by: Ankit Bisain, Eric Shen, Pitchayut Saengrungkongka, Sean Li

   Answer: $\boxed{72}$

   Solution: Let an arbitrary such number be $\overline{abcd}$. Then, we desire $11 \mid a+b+c+d$ and $11 \mid a-b+c-d$, where the latter comes from the well-known divisibility trick for 11. Sums and differences of multiples of 11 must also be multiples of 11, so this is equivalent to desiring $11 \mid a+c$ and $11 \mid b+d$.

   As $a \in [1, 9]$ and $b, c, d \in [0, 9]$, $a+c$ and $b+d$ must be either 0 or 11 (no larger multiple is achievable). There are 8 choices for such (a, c) and 9 choices for such (b, d), so the answer is $8 \cdot 9 = 72$.

4. Suppose that a and b are real numbers such that the line $y = ax + b$ intersects the graph of $y = x^2$ at two distinct points A and B. If the coordinates of the midpoint of AB are $(5, 101)$, compute $a + b$.

   Proposed by: Rishabh Das

   Answer: $\boxed{61}$

   Solution 1: Let $A = (r, r^2)$ and $B = (s, s^2)$. Since r and s are roots of $x^2 - ax - b$ with midpoint 5, $r + s = 10 = a$ (where the last equality follows by Vieta's formula).

   Now, as $-rs = b$ (Vieta's formula), observe that

   $$202 = r^2 + s^2 = (r + s)^2 - 2rs = 100 + 2b.$$

   This means $b = 51$, so the answer is $10 + 51 = 61$.

   Solution 2: As in the previous solution, let $A = (r, r^2)$ and $B = (s, s^2)$ and note $r + s = 10 = a$.

   Fixing $a = 10$, the y-coordinate of the midpoint is 50 when $b = 0$ (and changing b shifts the line up or down by its value). So, increasing b by 51 will make the midpoint have y-coordinate $50 + 51 = 101$, so the answer is $10 + 51 = 61$.

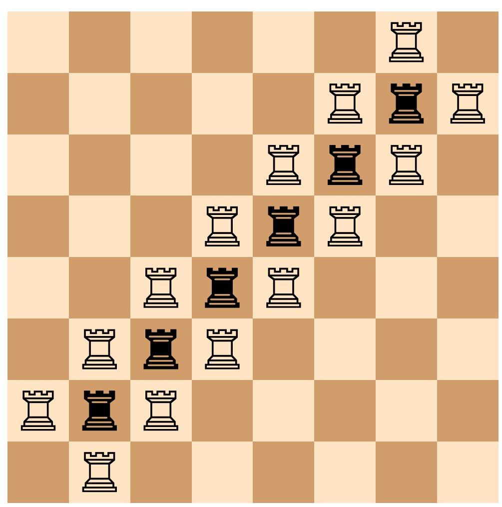
5. On an 8×8 chessboard, 6 black rooks and k white rooks are placed on different cells so that each rook only attacks rooks of the opposite color. Compute the maximum possible value of k.

   (Two rooks *attack* each other if they are in the same row or column and no rooks are between them.)

   Proposed by: Arul Kolla

   Answer: $\boxed{14}$

   Solution: The answer is $k = 14$. For a valid construction, place the black rooks on cells (a, a) for $2 \leq a \leq 7$ and the white rooks on cells $(a, a+1)$ and $(a+1, a)$ for $1 \leq a \leq 7$.



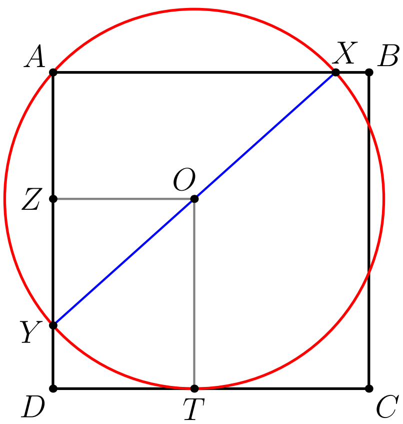
   Now, we prove the optimality. As rooks can only attack opposite color rooks, the color of rooks in each row is alternating. The difference between the number of black and white rooks is thus at most the number of rooks. Thus, $k \leq 6 + 8 = 14$.

6. Let $ABCD$ be a square of side length 5. A circle passing through A is tangent to segment CD at T and meets AB and AD again at $X \neq A$ and $Y \neq A$, respectively. Given that $XY = 6$, compute AT.

Proposed by: Maxim Li

Answer: $\boxed{\sqrt{30}}$

Solution:



Let O be the center of the circle, and let Z be the foot from O to AD. Since XY is a diameter, $OT = ZD = 3$, so $AZ = 2$. Then $OZ = \sqrt{5}$ and $AT = \sqrt{OZ^2 + 25} = \sqrt{30}$.

7. Compute all ordered triples (x, y, z) of real numbers satisfying the following system of equations:

$$xy + z = 40$$
$$xz + y = 51$$
$$x + y + z = 19.$$

Proposed by: Ethan Liu, Pitchayut Saengrungkongka

Answer: $\boxed{(12, 3, 4), (6, 5.4, 7.6)}$

Solution 1: By adding the first two equations, we can get

$$xy + z + xz + y = (x + 1)(y + z) = 91.$$

From the third equation we have

$$(x + 1) + (y + z) = 19 + 1 = 20,$$

so $x+1$ and $y+z$ are the two roots of $t^2 - 20t + 91 = 0$ by Vieta's theorem. As the quadratic equation can be decomposed into

$$(t - 7)(t - 13) = 0,$$

we know that either $x = 6, y + z = 13$ or $x = 12, y + z = 7$.

- If $x = 12$, by the first equation we have $12y+z = 40$, and substituting $y+z = 7$ we have $11y = 33$, $y = 3$ and $z = 4$.
- If $x = 6$, by the first equation we have $6y + z = 40$, and substituting $y + z = 13$ we have $5y = 27$, $y = 5.4$ and $z = 7.6$.

Hence, the two solutions are $(12, 3, 4)$ and $(6, 5.4, 7.6)$.

Solution 2: Viewing x as a constant, the equations become three linear equations in two variables y and z. This system can only have a solution if

$$\det \begin{bmatrix} x & 1 & 40 \\ 1 & x & 51 \\ 1 & 1 & 19-x \end{bmatrix} = 0.$$

Expanding out the determinant, we have

$$\begin{aligned} & x^2(19-x) + 51 + 40 - 51x - 40x - (19-x) = 0 \\ \implies & x^3 - 19x^2 + 90x - 72 = 0 \\ \implies & (x-1)(x^2 - 18x + 72) = 0 \\ \implies & (x-1)(x-6)(x-12) = 0 \end{aligned}$$

so $x = 1$, 6, or 12. If $x = 1$, the system has no solutions, and if $x = 6$ or 12, we can find y and z as in the first solution.

8. Mark writes the expression $\sqrt{d}$ for each positive divisor d of $8!$ on the board. Seeing that these expressions might not be worth points on HMMT, Rishabh simplifies each expression to the form $a\sqrt{b}$, where a and b are integers such that b is not divisible by the square of a prime number. (For example, $\sqrt{20}$, $\sqrt{16}$, and $\sqrt{6}$ simplify to $2\sqrt{5}$, $4\sqrt{1}$, and $1\sqrt{6}$, respectively.) Compute the sum of $a+b$ across all expressions that Rishabh writes.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{3480}$

Solution: Let $\sqrt{n}$ simplify to $a_n\sqrt{b_n}$. Notice that both a_n and b_n are multiplicative. Thus, $\sum_{d|n} a_d$ and $\sum_{d|n} b_d$ are multiplicative.

We consider the sum $\sum_{d|p^k} a_d$ and $\sum_{d|p^k} b_d$. Notice that for $d = p^l$, $a_d = p^{\lfloor l/2 \rfloor}$ and $b_d = p^{2\{l/2\}}$, so

$$\sum_{d|p^k} a_d = 2\left(\frac{p^{(k+1)/2} - 1}{p-1}\right) \qquad \text{and} \qquad \sum_{d|p^k} b_d = \frac{(p+1)(k+1)}{2}$$

for odd k, while

$$\sum_{d|p^k} a_d = \left(\frac{p^{(k+2)/2} + p^{k/2} - 2}{p-1}\right) \qquad \text{and} \qquad \sum_{d|p^k} b_d = \frac{(p+1)k}{2} + 1$$

for even k.

Notice $8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$, so

$$\sum_{d|8!} a_d = \left(\frac{2(16-1)}{2-1}\right)\left(\frac{9+3-2}{3-1}\right)(1+1)(1+1) = 30 \cdot 5 \cdot 2 \cdot 2 = 600$$

and

$$\sum_{d|8!} b_d = \left(\frac{3 \cdot 8}{2}\right)\left(1 + \frac{4 \cdot 2}{2}\right)(1+5)(1+7) = 12 \cdot 5 \cdot 6 \cdot 8 = 2880,$$

so the sum of $a_d + b_d$ would be $600 + 2880 = 3480$.

9. An entry in a grid is called a *saddle point* if it is the largest number in its row and the smallest number in its column. Suppose that each cell in a 3×3 grid is filled with a real number, each chosen independently and uniformly at random from the interval $[0, 1]$. Compute the probability that this grid has at least one saddle point.

   Proposed by: Benjamin Shimabukuro

   Answer: $\boxed{\frac{3}{10}}$

   Solution: With probability 1, all entries of the matrix are unique. If this is the case, we claim there can only be one saddle point. To see this, suppose A_{ij} and A_{kl} are both saddle points. They cannot be in the same row, since they cannot both be the greatest number in the same row, and similarly they cannot be in the same column, since they cannot both be the least number in the same column. If they are in different rows and different columns, then $A_{ij} < A_{il}$ and $A_{kl} > A_{il}$, so $A_{ij} < A_{kl}$. However, we also have $A_{ij} > A_{kj}$ and $A_{kl} < A_{kj}$, so $A_{ij} > A_{kl}$. This is a contradiction, so there is only one saddle point.

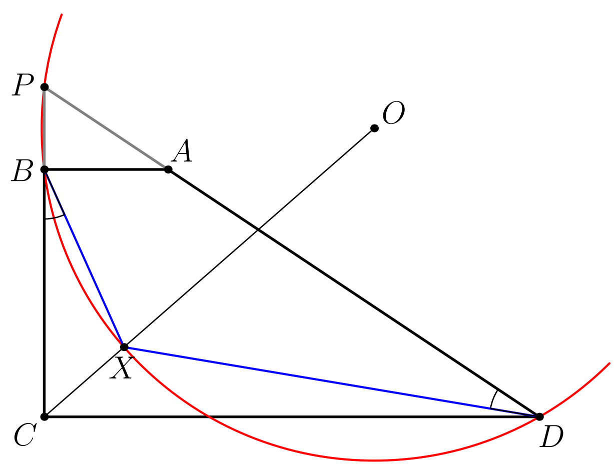
   Each entry of the matrix is equally likely to be a saddle point by symmetry, so we can just multiply the probability that A_{11} is a saddle point by 9 to find the answer. For A_{11} to be a saddle point, it must be greater than A_{21} and A_{31}, but less than A_{12} and A_{13}. There are $5! = 120$ equally likely ways that the numbers $A_{11}, A_{12}, A_{13}, A_{21}, A_{31}$ could be arranged in increasing order, and 4 of them work, so the probability that A_{11} is a saddle point is $\frac{1}{30}$. Therefore, the probability that A has a saddle point is $9 \cdot \frac{1}{30} = \frac{3}{10}$.

10. Let $ABCD$ be a convex trapezoid such that $\angle ABC = \angle BCD = 90°$, $AB = 3$, $BC = 6$, and $CD = 12$. Among all points X inside the trapezoid satisfying $\angle XBC = \angle XDA$, compute the minimum possible value of CX.

    Proposed by: Pitchayut Saengrungkongka

    Answer: $\boxed{\sqrt{113} - \sqrt{65}.}$

    Solution:



Let $P = AD \cap BC$. Then, the given angle condition $\angle XBC = \angle XAD$ implies that $\angle XBD + \angle XPD = 180°$, so X always lies on circle $\odot(PBD)$, which is fixed. Thus, we see that the locus if X is the arc $\widehat{BD}$ of $\odot(PBD)$. Let O and R be the center and the radius of $\odot(PBD)$. Then, by triangle inequality, we get that

$$CX \geq CO - OX = CO - R,$$

and the equality occurs when X is the intersection of segment CO and $\odot(PBD)$, as shown in the diagram above. Hence, the maximum value is $CO - R$.

To compute CO and R, we let T be the second intersection of $\odot(PBD)$ and CD. We can compute $BP = 2$, so by Power of Point, $CT \cdot CD = CP \cdot CB = 48$, so $CT = 4$, which means that $DT = 8$. The projections of O onto CD and CB are midpoints of BP and DT. Let those midpoints be M and N, respectively. Then, we get by Pythagorean theorem that

$$\begin{aligned} CO &= \sqrt{CN^2 + ON^2} = \sqrt{\left(4 + \tfrac{8}{2}\right)^2 + \left(6 + \tfrac{2}{2}\right)^2} = \sqrt{8^2 + 7^2} = \sqrt{113} \\ R &= \sqrt{BM^2 + MO^2} = \sqrt{1^2 + \left(4 + \tfrac{8}{2}\right)^2} = \sqrt{1^2 + 8^2} = \sqrt{65}, \end{aligned}$$

so the answer is $\sqrt{113} - \sqrt{65}$.