HMMT November 2023

November 11, 2023

Theme Round

1. Tyler has an infinite geometric series with sum 10. He increases the first term of his sequence by 4 and swiftly changes the subsequent terms so that the common ratio remains the same, creating a new geometric series with sum 15. Compute the common ratio of Tyler's series.

   Proposed by: Isabella Quan

   Answer: $\boxed{\frac{1}{5}}$

   Solution: Let a and r be the first term and common ratio of the original series, respectively. Then $\frac{a}{1-r} = 10$ and $\frac{a+4}{1-r} = 15$. Dividing these equations, we get that

$$\frac{a+4}{a} = \frac{15}{10} \implies a = 8.$$

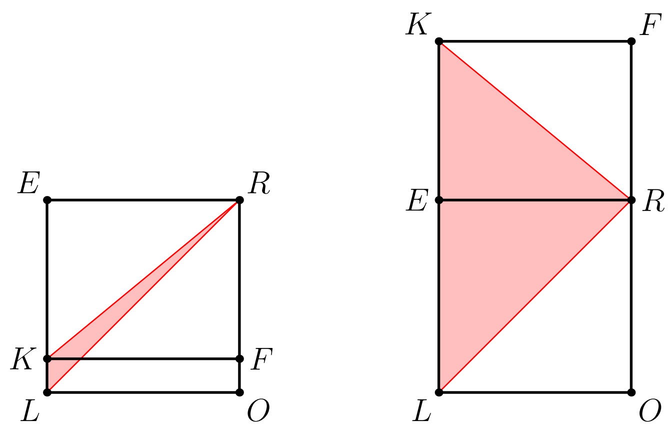
   Solving for r with $\frac{a}{1-r} = \frac{8}{1-r} = 10$ gives $r = \frac{1}{5}$.

2. Suppose rectangle $FOLK$ and square $LORE$ are on the plane such that $RL = 12$ and $RK = 11$. Compute the product of all possible areas of triangle RKL.

   Proposed by: Rishabh Das

   Answer: $\boxed{414}$

   Solution: There are two possible configurations, as shown below.



   If $RL = 12$, the side length of the square is $6\sqrt{2}$. Now

$$121 = RK^2 = RE^2 + EK^2 = (6\sqrt{2})^2 + EK^2,$$

   so $EK = 7$. Then the possible values of LK are $6\sqrt{2} \pm 7$. Note that the area of $\triangle RLK$ is

$$\frac{LK \cdot RE}{2} = LK \cdot 3\sqrt{2},$$

   and so the product of all possible areas are

$$\begin{aligned} 3\sqrt{2}\,(6\sqrt{2}+7) \cdot 3\sqrt{2}\,(6\sqrt{2}-7) &= (6\sqrt{2}+7)(6\sqrt{2}-7) \cdot (3\sqrt{2})^2 \\ &= (72-49) \cdot 18 = 414. \end{aligned}$$

3. There are 17 people at a party, and each has a reputation that is either 1, 2, 3, 4, or 5. Some of them split into pairs under the condition that within each pair, the two people's reputations differ by at most 1. Compute the largest value of k such that no matter what the reputations of these people are, they are able to form k pairs.

   Proposed by: Albert Wang

   Answer: $\boxed{7}$

   Solution: First, note that $k = 8$ fails when there are 15, 0, 1, 0, 1 people of reputation 1, 2, 3, 4, 5, respectively. This is because the two people with reputation 3 and 5 cannot pair with anyone, and there can only be at maximum $\lfloor \frac{15}{2} \rfloor = 7$ pairs of people with reputation 1.

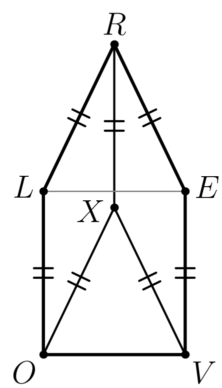
   Now, we show that $k = 7$ works. Suppose that we keep pairing people until we cannot make a pair anymore. Consider that moment. If there are two people with the same reputation, then these two people can pair up. Thus, there is at most one person for each reputation. Furthermore, if there are at least 4 people, then there must exist two people of consecutive reputations, so they can pair up. Thus, there are at most 3 people left, so we have formed at least $\frac{17-3}{2} = 7$ pairs.

4. Let $LOVER$ be a convex pentagon such that $LOVE$ is a rectangle. Given that $OV = 20$ and $LO = VE = RE = RL = 23$, compute the radius of the circle passing through R, O, and V.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{23}$

   Solution:



   Let X be the point such that $RXOL$ is a rhombus. Note that line RX defines a line of symmetry on the pentagon $LOVER$. Then by symmetry $RXVE$ is also a rhombus, so $RX = OX = VX = 23$. This makes X the center of the circle, and the radius is 23.

5. Compute the unique positive integer n such that $\frac{n^3-1989}{n}$ is a perfect square.

   Proposed by: Isaac Zhu

   Answer: $\boxed{13}$

   Solution 1: We need $n^2 - \frac{1989}{n}$ to be a perfect square, so $n \mid 1989$. Also, this perfect square would be less than n^2, so it would be at most $(n-1)^2 = n^2 - 2n + 1$. Thus,

$$\frac{1989}{n} \geq 2n - 1 \implies 1989 \geq 2n^2 - n,$$

   so $n \leq 31$. Moreover, we need

$$n^2 \geq \frac{1989}{n} \implies n^3 \geq 1989,$$

so $n \geq 13$. Factoring gives $1989 = 3^2 \cdot 13 \cdot 17$, which means the only possible values of n are 13 and 17. Checking both gives that only $n = 13$ works. (In fact, $\frac{13^3-1989}{13} = 4^2$.)

Solution 2: If $\frac{n^3-1989}{n} = d^2$ then $n^3 - nd^2 = 1989$. Factorizing gives

$$(n-d)n(n+d) = 3^2 \times 13 \times 17.$$

We can easily see that $n = 13$, $d = 4$ works since $1989 = 9 \times 13 \times 17$.

6. A function g is *ever more* than a function h if, for all real numbers x, we have $g(x) \geq h(x)$. Consider all quadratic functions $f(x)$ such that $f(1) = 16$ and $f(x)$ is ever more than both $(x+3)^2$ and x^2+9. Across all such quadratic functions f, compute the minimum value of $f(0)$.

Proposed by: Isabella Quan, Pitchayut Saengrungkongka, Alex Yi

Answer: $\boxed{\frac{21}{2}}$

Solution: Let $g(x) = (x+3)^2$ and $h(x) = x^2 + 9$. Then $f(1) = g(1) = 16$. Thus, $f(x) - g(x)$ has a root at $x = 1$. Since f is ever more than g, this means that in fact

$$f(x) - g(x) = c(x-1)^2$$

for some constant c.

Now

$$f(x) - h(x) = ((f(x) - g(x)) + (g(x) - h(x)) = c(x-1)^2 + 6x = cx^2 - (2c-6)x + c$$

is always nonnegative. The discriminant is

$$(2c-6)^2 - 4c^2 = 24c - 36 \geq 0,$$

so the smallest possible value of c is $\frac{3}{2}$. Then

$$f(0) = g(0) + c(x-1)^2 = 9 + c \geq \frac{21}{2},$$

with equality at $c = \frac{3}{2}$.

7. Betty has a 3×4 grid of dots. She colors each dot either red or maroon. Compute the number of ways Betty can color the grid such that there is no rectangle whose sides are parallel to the grid lines and whose vertices all have the same color.

Proposed by: Amy Feng

Answer: $\boxed{408}$

Solution: First suppose no 3 by 1 row is all red or all blue. Then each row is either two red and one blue, or two blue and one red. There are 6 possible configurations of such a row, and as long as no row is repeated, there's no monochromatic rectangle This gives $6 \cdot 5 \cdot 4 \cdot 3 = 360$ possibilities.

Now suppose we have a 3 by 1 row that's all red. Then the remaining rows must be two blue and one red, and all 3 such configurations must appear. This gives $4! = 24$, and having an all blue row is also $4! = 24$.

The final answer is $360 + 24 + 24 = 408$.

8. Call a number *feared* if it contains the digits 13 as a contiguous substring and *fearless* otherwise. (For example, 132 is feared, while 123 is fearless.) Compute the smallest positive integer n such that there exists a positive integer $a < 100$ such that n and $n + 10a$ are fearless while $n + a, n + 2a, \ldots, n + 9a$ are all feared.

   Proposed by: Rishabh Das

   Answer: $\boxed{1287}$

   Solution: First of all, note that we cannot have $n, n+a, \ldots, n+10a$ be less than 1000, since we cannot have fearless numbers have 13 as their last two digits since $a < 100$, and $129, 130, 131, \ldots, 139$ doesn't work as 139 is feared.

   Thus, we must utilize numbers of the form $13xy$, where 1, 3, x, and y are digits. If all of $n + a, n + 2a, \ldots, n + 9a$ start with 13, then $a \leq 12$, and the minimum we can achieve is 1288, with

$$1288, 1300, 1312, \ldots, 1384, 1396, 1408.$$

   If, however, $n + 9a = 1413$, then we can take $a = 14$ to get

$$1287, 1301, 1315, \ldots, 1399, 1413, 1427,$$

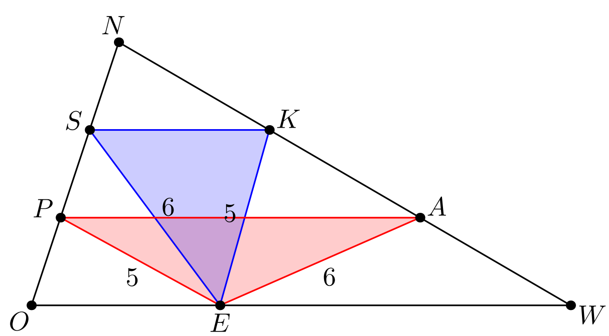
   so the minimum possible value is 1287.

9. Pentagon $SPEAK$ is inscribed in triangle NOW such that S and P lie on segment NO, K and A lie on segment NW, and E lies on segment OW. Suppose that $NS = SP = PO$ and $NK = KA = AW$. Given that $EP = EK = 5$ and $EA = ES = 6$, compute OW.

   Proposed by: Rishabh Das

   Answer: $\boxed{\frac{3\sqrt{610}}{5} = 3\sqrt{\frac{122}{5}}}$

   Solution:



   Note that $[ESK] = [EPA]$, since one has half the base but double the height. Since the sides are the same, we must have $\sin \angle SEK = \sin \angle PEA$, so $\angle SEK + \angle PEA = 180°$.

   Let $OW = 3x$, so $SK = x$ and $PA = 2x$. Then by the law of cosines

$$x^2 = 61 - 60 \cos \angle SEK$$
$$4x^2 = 61 - 60 \cos \angle PEA.$$

   Summing these two gives $5x^2 = 122$, since $\cos \angle SEK = -\cos \angle PEA$. Then $x = \sqrt{\frac{122}{5}}$, which means $3x = \frac{3\sqrt{610}}{5}$.

10. It is midnight on April 29th, and Abigail is listening to a song by her favorite artist while staring at her clock, which has an hour, minute, and second hand. These hands move continuously. Between two consecutive midnights, compute the number of times the hour, minute, and second hands form two equal angles and no two hands overlap.

Proposed by: Evan Erickson

Answer: $\boxed{5700}$

Solution: Let $t \in [0,2]$ represent the position of the hour hand, i.e., how many full revolutions it has made. Then, the position of the minute hand is $12t$ (it makes 12 full revolutions per 1 revolution of the hour hand), and the position of the second hand is $720t$ (it makes 60 full revolutions per 1 revolution of the minute hand). Then, in order for equal angles to be formed, we need $(a-b)-(b-c) = a-2b+c \equiv 0 \pmod 1$, where a, b, c is a permutation of $t, 12t, 720t$. (Here, b would correspond to the hand that's the angle bisector.) Checking all three possibilities,

$$\begin{aligned} 12t - 2(t) + 720t \equiv 697t &\equiv 0 \pmod 1, \\ t - 2(12t) + 720t \equiv 730t &\equiv 0 \pmod 1, \\ t - 2(720t) + 12t \equiv -1427t &\equiv 0 \pmod 1. \end{aligned}$$

Then we require t to be a multiple of $\frac{1}{697}$, $\frac{1}{730}$, or $\frac{1}{1427}$. Since 697, 730, and 1427 are pairwise relatively prime, the possible values of t are

$$\begin{aligned} &\frac{1}{697}, \frac{2}{697}, \ldots, \frac{696}{697}, \frac{698}{697}, \ldots, \frac{2 \cdot 697 - 1}{697}, \\ &\frac{1}{730}, \frac{2}{730}, \ldots, \frac{729}{730}, \frac{731}{730}, \ldots, \frac{2 \cdot 730 - 1}{730}, \\ &\frac{1}{1427}, \frac{2}{1427}, \ldots, \frac{1426}{1427}, \frac{1428}{1427}, \ldots, \frac{2 \cdot 1427 - 1}{1427} \end{aligned}$$

since $t \in [0,2]$. This gives a count of $2((697-1)+(730-1)+(1427-1)) = 5702$.

Note that in the above count we don't count $t = 0, 1, 2$ since then all three hands would overlap. If two hands overlap, then one of $11t, 708t, 719t \equiv 0 \pmod 1$, and the only way one of these can happen and t being a multiple of $\frac{1}{697}$, $\frac{1}{730}$, or $\frac{1}{1427}$ is if $t = \frac{1}{2}$ and $t = \frac{3}{2}$ (which correspond to 6:00 AM and PM). This is because the only pair of numbers that are not relatively prime among 11, 708, 719, 697, 730, 1427 is 708 and 730. The only common divisor of these two numbers is 2, hence $t = \frac{1}{2}, \frac{3}{2}$. Thus the final answer is $5702 - 2 = 5700$.