HMMT February 2024

February 17, 2024

Algebra and Number Theory Round

1. Suppose r, s, and t are nonzero reals such that the polynomial $x^2 + rx + s$ has s and t as roots, and the polynomial $x^2 + tx + r$ has 5 as a root. Compute s.

2. Suppose a and b are positive integers. Isabella and Vidur both fill up an $a \times b$ table. Isabella fills it up with numbers $1, 2, \ldots, ab$, putting the numbers $1, 2, \ldots, b$ in the first row, $b+1, b+2, \ldots, 2b$ in the second row, and so on. Vidur fills it up like a multiplication table, putting ij in the cell in row i and column j. (Examples are shown for a 3×4 table below.)

1	2	3	4
5	6	7	8
9	10	11	12

Isabella's Grid

1	2	3	4
2	4	6	8
3	6	9	12

Vidur's Grid

Isabella sums up the numbers in her grid, and Vidur sums up the numbers in his grid; the difference between these two quantities is 1200. Compute $a + b$.

3. Compute the sum of all two-digit positive integers x such that for all three-digit (base 10) positive integers $\underline{a}\,\underline{b}\,\underline{c}$, if $\underline{a}\,\underline{b}\,\underline{c}$ is a multiple of x, then the three-digit (base 10) number $\underline{b}\,\underline{c}\,\underline{a}$ is also a multiple of x.

4. Let $f(x)$ be a quotient of two quadratic polynomials. Given that $f(n) = n^3$ for all $n \in \{1, 2, 3, 4, 5\}$, compute $f(0)$.

5. Compute the unique ordered pair (x, y) of real numbers satisfying the system of equations

$$\frac{x}{\sqrt{x^2+y^2}} - \frac{1}{x} = 7 \quad \text{and} \quad \frac{y}{\sqrt{x^2+y^2}} + \frac{1}{y} = 4.$$

6. Compute the sum of all positive integers n such that $50 \le n \le 100$ and $2n + 3$ does not divide $2^{n!} - 1$.

7. Let $P(n) = (n - 1^3)(n - 2^3) \ldots (n - 40^3)$ for positive integers n. Suppose that d is the largest positive integer that divides $P(n)$ for every integer $n > 2023$. If d is a product of m (not necessarily distinct) prime numbers, compute m.

8. Let $\zeta = \cos\frac{2\pi}{13} + i \sin\frac{2\pi}{13}$. Suppose $a > b > c > d$ are positive integers satisfying

$$|\zeta^a + \zeta^b + \zeta^c + \zeta^d| = \sqrt{3}.$$

Compute the smallest possible value of $1000a + 100b + 10c + d$.

9. Suppose a, b, and c are complex numbers satisfying

$$\begin{aligned} a^2 &= b - c, \\ b^2 &= c - a, \text{ and} \\ c^2 &= a - b. \end{aligned}$$

Compute all possible values of $a + b + c$.

10. A polynomial $f \in \mathbb{Z}[x]$ is called *splitty* if and only if for every prime p, there exist polynomials $g_p, h_p \in \mathbb{Z}[x]$ with $\deg g_p, \deg h_p < \deg f$ and all coefficients of $f - g_p h_p$ are divisible by p. Compute the sum of all positive integers $n \le 100$ such that the polynomial $x^4 + 16x^2 + n$ is splitty.