HMMT February 2024

February 17, 2024

Algebra and Number Theory Round

1. Suppose r, s, and t are nonzero reals such that the polynomial $x^2 + rx + s$ has s and t as roots, and the polynomial $x^2 + tx + r$ has 5 as a root. Compute s.

   Proposed by: Rishabh Das

   Answer: $\boxed{29}$

   Solution: The first equation implies $st = s$, so $t = 1$. Then $x^2 + x + r$ has 5 as a root, so $r + 30 = 0$, implying $r = -30$. Finally, $x^2 - 30x + s$ has 1 as a root, so $s = \boxed{29}$.

   Remark: We missed the case of $s = t$, so $x^2 + rx + s$ has $s = t$ as one root, and 1 as the other root (by Vieta's). This means $r = -s - 1$. Then

$$x^2 + tx + r = x^2 + sx - (s + 1) = (x + (s + 1))(x - 1)$$

   has 5 as a root, so $s = -6$ is another solution. During the competition, both the answers -6 and 29 (as well as "29 or -6") were accepted.

2. Suppose a and b are positive integers. Isabella and Vidur both fill up an $a \times b$ table. Isabella fills it up with numbers $1, 2, \ldots, ab$, putting the numbers $1, 2, \ldots, b$ in the first row, $b + 1, b + 2, \ldots, 2b$ in the second row, and so on. Vidur fills it up like a multiplication table, putting ij in the cell in row i and column j. (Examples are shown for a 3×4 table below.)

1	2	3	4
5	6	7	8
9	10	11	12

Isabella's Grid

1	2	3	4
2	4	6	8
3	6	9	12

Vidur's Grid

   Isabella sums up the numbers in her grid, and Vidur sums up the numbers in his grid; the difference between these two quantities is 1200. Compute $a + b$.

   Proposed by: Rishabh Das

   Answer: $\boxed{21}$

   Solution: Using the formula $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, we get

$$\begin{aligned} \frac{ab(ab+1)}{2} - \frac{a(a+1)}{2} \cdot \frac{b(b+1)}{2} &= \frac{ab\left(2(ab+1) - (a+1)(b+1)\right)}{4} \\ &= \frac{ab(ab - a - b + 1)}{4} \\ &= \frac{ab(a-1)(b-1)}{4} \\ &= \frac{a(a-1)}{2} \cdot \frac{b(b-1)}{2}. \end{aligned}$$

   This means we can write the desired equation as

$$a(a-1) \cdot b(b-1) = 4800.$$

   Assume $b \le a$, so we know $b(b-1) \le a(a-1)$, so $b(b-1) < 70$. Thus, $b \le 8$.

   If $b = 7$ or $b = 8$, then $b(b-1)$ has a factor of 7, which 4800 does not, so $b \le 6$.

If $b = 6$ then $b(b-1) = 30$, so $a(a-1) = 160$, which can be seen to have no solutions.

If $b = 5$ then $b(b-1) = 20$, so $a(a-1) = 240$, which has the solution $a = 16$, giving $5 + 16 = \boxed{21}$.

We need not continue since we are guaranteed only one solution, but we check the remaining cases for completeness. If $b = 4$ then $a(a-1) = \frac{4800}{12} = 400$, which has no solutions. If $b = 3$ then $a(a-1) = \frac{4800}{6} = 800$ which has no solutions. Finally, if $b = 2$ then $a(a-1) = \frac{4800}{2} = 2400$, which has no solutions.

The factorization of the left side may come as a surprise; here's a way to see it should factor without doing the algebra. If either $a = 1$ or $b = 1$, then the left side simplifies to 0. As a result, both $a - 1$ and $b - 1$ should be a factor of the left side.

3. Compute the sum of all two-digit positive integers x such that for all three-digit (base 10) positive integers $\underline{a}\,\underline{b}\,\underline{c}$, if $\underline{a}\,\underline{b}\,\underline{c}$ is a multiple of x, then the three-digit (base 10) number $\underline{b}\,\underline{c}\,\underline{a}$ is also a multiple of x.

   Proposed by: Karthik Venkata Vedula

   Answer: $\boxed{64}$

   Solution: Note that $\overline{abc0} - \overline{bca} = a(10^4 - 1)$ must also be a multiple of x. Choosing $a = 1$ means that x divides $10^3 - 1$, and this is clearly a necessary and sufficient condition. The only two-digit factors of $10^3 - 1$ are 27 and 37, so our answer is $27 + 37 = \boxed{64}$.

4. Let $f(x)$ be a quotient of two quadratic polynomials. Given that $f(n) = n^3$ for all $n \in \{1, 2, 3, 4, 5\}$, compute $f(0)$.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{\frac{24}{17}}$

   Solution: Let $f(x) = p(x)/q(x)$. Then, $x^3 q(x) - p(x)$ has $1, 2, 3, 4, 5$ as roots. Therefore, WLOG, let

$$x^3 q(x) - p(x) = (x-1)(x-2)(x-3)(x-4)(x-5) = x^5 - 15x^4 + 85x^3 - \dots$$

   Thus, $q(x) = x^2 - 15x + 85$, so $q(0) = 85$. Plugging $x = 0$ in the above equation also gives $-p(0) = -120$. Hence, the answer is $\frac{120}{85} = \boxed{\frac{24}{17}}$.

   Remark. From the solution above, it is not hard to see that the unique f that satisfies the problem is

$$f(x) = \frac{225x^2 - 274x + 120}{x^2 - 15x + 85}.$$

5. Compute the unique ordered pair (x, y) of real numbers satisfying the system of equations

$$\frac{x}{\sqrt{x^2 + y^2}} - \frac{1}{x} = 7 \quad \text{and} \quad \frac{y}{\sqrt{x^2 + y^2}} + \frac{1}{y} = 4.$$

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{\left(-\frac{13}{96}, \frac{13}{40}\right)}$

   Solution 1: Consider vectors

$$\begin{pmatrix} x/\sqrt{x^2+y^2} \\ y/\sqrt{x^2+y^2} \end{pmatrix} \text{ and } \begin{pmatrix} -1/x \\ 1/y \end{pmatrix}.$$

They are orthogonal and add up to $\begin{pmatrix}7\\4\end{pmatrix}$, which have length $\sqrt{7^2+4^2} = \sqrt{65}$. The first vector has length 1, so by Pythagorean's theorem, the second vector has length $\sqrt{65-1} = 8$, so we have

$$\frac{1}{x^2} + \frac{1}{y^2} = 64 \implies \sqrt{x^2+y^2} = \pm 8xy.$$

However, the first equation indicates that $x < 0$, while the second equation indicates that $y > 0$, so $xy < 0$. Thus, $\sqrt{x^2+y^2} = -8xy$. Plugging this into both of the starting equations give

$$-\frac{1}{8y} - \frac{1}{x} = 7 \text{ and } -\frac{1}{8x} + \frac{1}{y} = 4.$$

Solving this gives $(x, y) = \boxed{\left(-\frac{13}{96}, \frac{13}{40}\right)}$, which works.

Solution 2: Let $x = r\cos\theta$ and $y = r\sin\theta$. Then our equations read

$$\begin{aligned}\cos\theta - \frac{1}{r\cos\theta} &= 7\\ \sin\theta + \frac{1}{r\sin\theta} &= 4.\end{aligned}$$

Multiplying the first equation by $\cos\theta$ and the second by $\sin\theta$, and then adding the two gives $7\cos\theta + 4\sin\theta = 1$. This means

$$4\sin\theta = 1 - 7\cos\theta \implies 16\sin^2\theta = 1 - 14\cos\theta + 49\cos^2\theta \implies 65\cos^2\theta - 14\cos\theta - 15 = 0.$$

This factors as $(13\cos\theta + 5)(5\cos\theta - 3) = 0$, so $\cos\theta$ is either $\frac{3}{5}$ or $-\frac{5}{13}$. This means either $\cos\theta = \frac{3}{5}$ and $\sin\theta = -\frac{4}{5}$, or $\cos\theta = -\frac{5}{13}$ and $\sin\theta = \frac{12}{13}$.

The first case, plugging back in, makes r a negative number, a contradiction, so we take the second case. Then $x = \frac{1}{\cos\theta - 7} = -\frac{13}{96}$ and $y = \frac{1}{4 - \sin\theta} = \frac{13}{40}$. The answer is $(x, y) = \boxed{\left(-\frac{13}{96}, \frac{13}{40}\right)}$.

6. Compute the sum of all positive integers n such that $50 \le n \le 100$ and $2n+3$ does not divide $2^{n!} - 1$.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{222}$

   Solution: We claim that if $n \ge 10$, then $2n+3 \nmid 2^{n!} - 1$ if and only if both $n+1$ and $2n+3$ are prime.

   If both $n+1$ and $2n+3$ are prime, then assume $2n+3 \mid 2^{n!} - 1$. By Fermat Little Theorem, $2n+3 \mid 2^{2n+2} + 1$. However, since $n+1$ is prime, $\gcd(2n+2, n!) = 2$, so $2n+3 \mid 2^2 - 1 = 3$, a contradiction.

   If $2n+3$ is composite, then $\varphi(2n+3)$ is even and is at most $2n$, so $\varphi(2n+3) \mid n!$, done.

   If $n+1$ is composite but $2n+3$ is prime, then $2n+2 \mid n!$, so $2n+3 \mid 2^{n!} - 1$.

   The prime numbers between 50 and 100 are $53, 59, 61, 67, 71, 73, 79, 83, 89, 97$. If one of these is $n+1$, then the only numbers that make $2n+3$ prime are $53, 83$, and 89, making n one of 52, 82, and 88. These sum to $\boxed{222}$.

7. Let $P(n) = (n - 1^3)(n - 2^3)\dots(n - 40^3)$ for positive integers n. Suppose that d is the largest positive integer that divides $P(n)$ for every integer $n > 2023$. If d is a product of m (not necessarily distinct) prime numbers, compute m.

   Proposed by: Nithid Anchaleenukoon

   Answer: $\boxed{48}$

Solution: We first investigate what primes divide d. Notice that a prime p divides $P(n)$ for all $n \geq 2024$ if and only if $\{1^3, 2^3, \ldots, 40^3\}$ contains all residues in modulo p. Hence, $p \leq 40$. Moreover, $x^3 \equiv 1$ must not have other solution in modulo p than 1, so $p \not\equiv 1 \pmod 3$. Thus, the set of prime divisors of d is $S = \{2, 3, 5, 11, 17, 23, 29\}$.

Next, the main claim is that for all prime $p \in S$, the minimum value of $\nu_p(P(n))$ across all $n \geq 2024$ is $\left\lfloor \frac{40}{p} \right\rfloor$. To see why, note the following:

- **Lower Bound.** Note that for all $n \in \mathbb{Z}$, one can group $n - 1^3$, $n - 2^3$, $\ldots$, $n - 40^3$ into $\left\lfloor \frac{40}{p} \right\rfloor$ contiguous blocks of size p. Since $p \not\equiv 1 \pmod 3$, x^3 span through all residues modulo p, so each block will have one number divisible by p. Hence, among $n - 1^3$, $n - 2^3$, $\ldots$, $n - 40^3$, at least $\left\lfloor \frac{40}{p} \right\rfloor$ are divisible by p, implying that $\nu_p(P(n)) > \left\lfloor \frac{40}{p} \right\rfloor$.
- **Upper Bound.** We pick any n such that $\nu_p(n) = 1$ so that only terms in form $n - p^3$, $n - (2p)^3$, $\ldots$ are divisible by p. Note that these terms are not divisible by p^2 either, so in this case, we have $\nu_p(P(n)) = \left\lfloor \frac{40}{p} \right\rfloor$.

Hence, $\nu_p(d) = \left\lfloor \frac{40}{p} \right\rfloor$ for all prime $p \in S$. Thus, the answer is

$$\sum_{p \in S} \left\lfloor \frac{40}{p} \right\rfloor = \left\lfloor \frac{40}{2} \right\rfloor + \left\lfloor \frac{40}{3} \right\rfloor + \left\lfloor \frac{40}{5} \right\rfloor + \left\lfloor \frac{40}{11} \right\rfloor + \left\lfloor \frac{40}{17} \right\rfloor + \left\lfloor \frac{40}{23} \right\rfloor + \left\lfloor \frac{40}{29} \right\rfloor = \boxed{48}.$$

8. Let $\zeta = \cos \frac{2\pi}{13} + i \sin \frac{2\pi}{13}$. Suppose $a > b > c > d$ are positive integers satisfying

$$|\zeta^a + \zeta^b + \zeta^c + \zeta^d| = \sqrt{3}.$$

Compute the smallest possible value of $1000a + 100b + 10c + d$.

Proposed by: Rishabh Das

Answer: $\boxed{7521}$

Solution: We may as well take $d = 1$ and shift the other variables down by d to get $|\zeta^{a'} + \zeta^{b'} + \zeta^{c'} + 1| = \sqrt{3}$. Multiplying by its conjugate gives

$$(\zeta^{a'} + \zeta^{b'} + \zeta^{c'} + 1)(\zeta^{-a'} + \zeta^{-b'} + \zeta^{-c'} + 1) = 3.$$

Expanding, we get

$$1 + \sum_{x, y \in S,\, x \neq y} \zeta^{x-y} = 0,$$

where $S = \{a', b', c', 0\}$.

This is the sum of 13 terms, which hints that $S - S$ should form a complete residue class mod 13. We can prove this with the fact that the minimal polynomial of ζ is $1 + x + x^2 + \cdots + x^{12}$.

The minimum possible value of a' is 6, as otherwise every difference would be between -5 and 5 mod 13. Take $a' = 6$. If $b' \leq 2$ then we couldn't form a difference of 3 in S, so $b' \geq 3$. Moreover, $6 - 3 = 3 - 0$, so $3 \notin S$, so $b' = 4$ is the best possible. Then $c' = 1$ works.

If $a' = 6$, $b' = 4$, and $c' = 1$, then $a = 7$, $b = 5$, $c = 2$, and $d = 1$, so the answer is $\boxed{7521}$.

9. Suppose a, b, and c are complex numbers satisfying

$$\begin{aligned} a^2 &= b - c, \\ b^2 &= c - a, \text{ and} \\ c^2 &= a - b. \end{aligned}$$

Compute all possible values of $a + b + c$.

Proposed by: Rishabh Das

Answer: $\boxed{0, \pm i\sqrt{6}}$

Solution: Summing the equations gives $a^2 + b^2 + c^2 = 0$ and summing a times the first equation and etc. gives $a^3 + b^3 + c^3 = 0$. Let $a + b + c = k$. Then $a^2 + b^2 + c^2 = 0$ means $ab + bc + ca = k^2/2$, and

$$a^3 + b^3 + c^3 = 0 \implies -3abc = a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = -k^3/2,$$

so $abc = k^3/6$.

This means a, b, and c are roots of the cubic

$$x^3 - kx^2 + (k^2/2)x - (k^3/6) = 0$$

for some k.

Next, note that

$$\begin{aligned} a^4 + b^4 + c^4 &= \sum_{\text{cyc}} a(ka^2 - (k^2/2)a + (k^3/6)) \\ &= \sum_{\text{cyc}} k(ka^2 - (k^2/2)a + (k^3/6)) - (k^2/2)a^2 + (k^3/6)a \\ &= \sum_{\text{cyc}} (k^2/2)a^2 - (k^3/3)a + (k^4/6) \\ &= -k^4/3 + k^4/2 \\ &= k^4/6. \end{aligned}$$

After this, there are two ways to extract the values of k.

- Summing squares of each equation gives

  $$a^4 + b^4 + c^4 = \sum_{\text{cyc}} (a - b)^2 = 2(a^2 + b^2 + c^2) - 2(ab + bc + ca) = -k^2,$$

  so

  $$\frac{k^4}{6} = -k^2 \implies k = \boxed{0, \pm i\sqrt{6}}.$$

- Summing a^2 times the first equation, etc. gives

  $$a^4 + b^4 + c^4 = \sum_{\text{cyc}} a^2(b - c) = -(a - b)(b - c)(c - a) = -a^2b^2c^2 = -\frac{k^6}{36},$$

  so

  $$\frac{k^4}{6} = -\frac{k^6}{36} \implies k = \boxed{0, \pm i\sqrt{6}}.$$

We can achieve $k = 0$ with $a = b = c = 0$. Letting a, b, and c be the roots of $x^3 - (i\sqrt{6})x^2 - 3x + (i\sqrt{6})$ will force one of $a^2 = b - c$ and all other equalities or $a^2 - c - b$ and all other equalities to hold, if the latter happens, swap b and c. Finally, for these (a, b, c), take $(-a, -c, -b)$ to get $-i\sqrt{6}$. Thus, all of these are achievable.

10. A polynomial $f \in \mathbb{Z}[x]$ is called *splitty* if and only if for every prime p, there exist polynomials $g_p, h_p \in \mathbb{Z}[x]$ with $\deg g_p, \deg h_p < \deg f$ and all coefficients of $f - g_p h_p$ are divisible by p. Compute the sum of all positive integers $n \leq 100$ such that the polynomial $x^4 + 16x^2 + n$ is splitty.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{693}$

Solution: We claim that $x^4 + ax^2 + b$ is splitty if and only if either b or $a^2 - 4b$ is a perfect square. (The latter means that the polynomial splits into $(x^2 - r)(x^2 - s)$).

Assuming the characterization, one can easily extract the answer. For $a = 16$ and $b = n$, one of n and $64 - n$ has to be a perfect square. The solutions to this that are at most 64 form 8 pairs that sum to 64 (if we include 0), and then we additionally have 81 and 100. This means the sum is $64 \cdot 8 + 81 + 100 = \boxed{693}$.

Now, we move on to prove the characterization.

Necessity.

Take a prime p such that neither $a^2 - 4b$ nor b is a quadratic residue modulo p (exists by Dirichlet + CRT + QR). Work in $\mathbb{F}_p$. Now, suppose that

$$x^4 + ax^2 + b = (x^2 + mx + n)(x^2 + sx + t).$$

Then, looking at the x^3-coefficient gives $m + s = 0$ or $s = -m$. Looking at the x-coefficient gives $m(n - t) = 0$.

- If $m = 0$, then $s = 0$, so $x^4 + ax^2 + b = (x^2 + n)(x^2 + t)$, which means $a^2 - 4b = (n + t)^2 - 4nt = (n - t)^2$, a quadratic residue modulo p, contradiction.
- If $n = t$, then $b = nt$ is a square modulo p, a contradiction. (The major surprise of this problem is that this suffices, which will be shown below.)

Sufficiency.

Clearly, the polynomial splits in $p = 2$ because in $\mathbb{F}_2[x]$, we have $x^4 + ax^2 + b = (x^2 + ax + b)^2$. Now, assume p is odd.

If $a^2 - 4b$ is a perfect square, then $x^4 + ax^2 + b$ splits into $(x^2 - r)(x^2 - s)$ even in $\mathbb{Z}[x]$.

If b is a perfect square, then let $b = k^2$. We then note that

- $x^4 + ax^2 + b$ splits in form $(x^2 - r)(x^2 - s)$ if $\left(\frac{a^2 - 4k^2}{p}\right) = 1$.
- $x^4 + ax^2 + b$ splits in form $(x^2 + rx + k)(x^2 - rx + k)$ if $a = 2k - r^2$, or $\left(\frac{2k - a}{p}\right) = 1$.
- $x^4 + ax^2 + b$ splits in form $(x^2 + rx - k)(x^2 - rx - k)$ if $a = -2k - r^2$, or $\left(\frac{-2k - a}{p}\right) = 1$.

Since $(2k - a)(-2k - a) = a^2 - 4k^2$, it follows that at least one of these must happen.