HMMT February 2024
February 17, 2024
Combinatorics Round

1. Compute the number of ways to divide a 20×24 rectangle into 4×5 rectangles. (Rotations and reflections are considered distinct.)

2. A *lame king* is a chess piece that can move from a cell to any cell that shares at least one vertex with it, except for the cells in the same column as the current cell.

   A lame king is placed in the top-left cell of a 7×7 grid. Compute the maximum number of cells it can visit without visiting the same cell twice (including its starting cell).

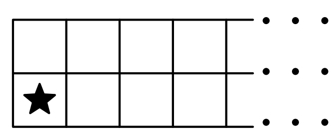
3. Compute the number of ways there are to assemble 2 red unit cubes and 25 white unit cubes into a $3 \times 3 \times 3$ cube such that red is visible on exactly 4 faces of the larger cube. (Rotations and reflections are considered distinct.)

4. Sally the snail sits on the 3×24 lattice of points (i, j) for all $1 \leq i \leq 3$ and $1 \leq j \leq 24$. She wants to visit every point in the lattice exactly once. In a move, Sally can move to a point in the lattice exactly one unit away. Given that Sally starts at $(2, 1)$, compute the number of possible paths Sally can take.

5. The country of HMMTLand has 8 cities. Its government decides to construct several two-way roads between pairs of distinct cities. After they finish construction, it turns out that each city can reach exactly 3 other cities via a single road, and from any pair of distinct cities, either exactly 0 or 2 other cities can be reached from both cities by a single road. Compute the number of ways HMMTLand could have constructed the roads.

6. In each cell of a 4×4 grid, one of the two diagonals is drawn uniformly at random. Compute the probability that the resulting 32 triangular regions can be colored red and blue so that any two regions sharing an edge have different colors.

7. There is a grid of height 2 stretching infinitely in one direction. Between any two edge-adjacent cells of the grid, there is a door that is locked with probability $\frac{1}{2}$ independent of all other doors. Philip starts in a corner of the grid (in the starred cell). Compute the expected number of cells that Philip can reach, assuming he can only travel between cells if the door between them is unlocked.



8. Rishabh has 2024 pairs of socks in a drawer. He draws socks from the drawer uniformly at random, without replacement, until he has drawn a pair of identical socks. Compute the expected number of unpaired socks he has drawn when he stops.

9. Compute the number of triples (f, g, h) of permutations on $\{1, 2, 3, 4, 5\}$ such that

$$\begin{aligned} f(g(h(x))) &= h(g(f(x))) = g(x), \\ g(h(f(x))) &= f(h(g(x))) = h(x), \text{ and} \\ h(f(g(x))) &= g(f(h(x))) = f(x) \end{aligned}$$

   for all $x \in \{1, 2, 3, 4, 5\}$.

10. A *peacock* is a ten-digit positive integer that uses each digit exactly once. Compute the number of peacocks that are exactly twice another peacock.