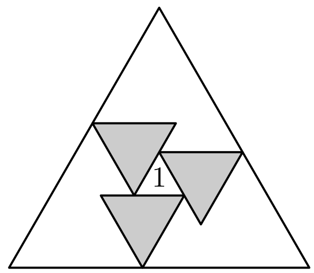
HMMT February 2024

February 17, 2024

Geometry Round

1. Inside an equilateral triangle of side length 6, three congruent equilateral triangles of side length x with sides parallel to the original equilateral triangle are arranged so that each has a vertex on a side of the larger triangle, and a vertex on another one of the three equilateral triangles, as shown below.



A smaller equilateral triangle formed between the three congruent equilateral triangles has side length 1. Compute x.

2. Let ABC be a triangle with $\angle BAC = 90^\circ$. Let D, E, and F be the feet of altitude, angle bisector, and median from A to BC, respectively. If $DE = 3$ and $EF = 5$, compute the length of BC.

3. Let Ω and ω be circles with radii 123 and 61, respectively, such that the center of Ω lies on ω. A chord of Ω is cut by ω into three segments, whose lengths are in the ratio $1 : 2 : 3$ in that order. Given that this chord is not a diameter of Ω, compute the length of this chord.

4. Let $ABCD$ be a square, and let ℓ be a line passing through the midpoint of segment $\overline{AB}$ that intersects segment $\overline{BC}$. Given that the distances from A and C to ℓ are 4 and 7, respectively, compute the area of $ABCD$.

5. Let $ABCD$ be a convex trapezoid such that $\angle DAB = \angle ABC = 90^\circ$, $DA = 2$, $AB = 3$, and $BC = 8$. Let ω be a circle passing through A and tangent to segment $\overline{CD}$ at point T. Suppose that the center of ω lies on line BC. Compute CT.

6. In triangle ABC, a circle ω with center O passes through B and C and intersects segments $\overline{AB}$ and $\overline{AC}$ again at B' and C', respectively. Suppose that the circles with diameters BB' and CC' are externally tangent to each other at T. If $AB = 18$, $AC = 36$, and $AT = 12$, compute AO.

7. Let ABC be an acute triangle. Let D, E, and F be the feet of altitudes from A, B, and C to sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, and let Q be the foot of altitude from A to line EF. Given that $AQ = 20$, $BC = 15$, and $AD = 24$, compute the perimeter of triangle DEF.

8. Let $ABTCD$ be a convex pentagon with area 22 such that $AB = CD$ and the circumcircles of triangles TAB and TCD are internally tangent. Given that $\angle ATD = 90^\circ$, $\angle BTC = 120^\circ$, $BT = 4$, and $CT = 5$, compute the area of triangle TAD.

9. Let ABC be a triangle. Let X be the point on side $\overline{AB}$ such that $\angle BXC = 60^\circ$. Let P be the point on segment $\overline{CX}$ such that $BP \perp AC$. Given that $AB = 6$, $AC = 7$, and $BP = 4$, compute CP.

10. Suppose point P is inside quadrilateral $ABCD$ such that

$$\begin{aligned}
\angle PAB &= \angle PDA, \\
\angle PAD &= \angle PDC, \\
\angle PBA &= \angle PCB, \text{ and} \\
\angle PBC &= \angle PCD.
\end{aligned}$$

If $PA = 4$, $PB = 5$, and $PC = 10$, compute the perimeter of $ABCD$.