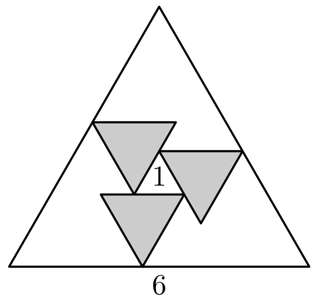
HMMT February 2024

February 17, 2024

Geometry Round

1. Inside an equilateral triangle of side length 6, three congruent equilateral triangles of side length x with sides parallel to the original equilateral triangle are arranged so that each has a vertex on a side of the larger triangle, and a vertex on another one of the three equilateral triangles, as shown below.

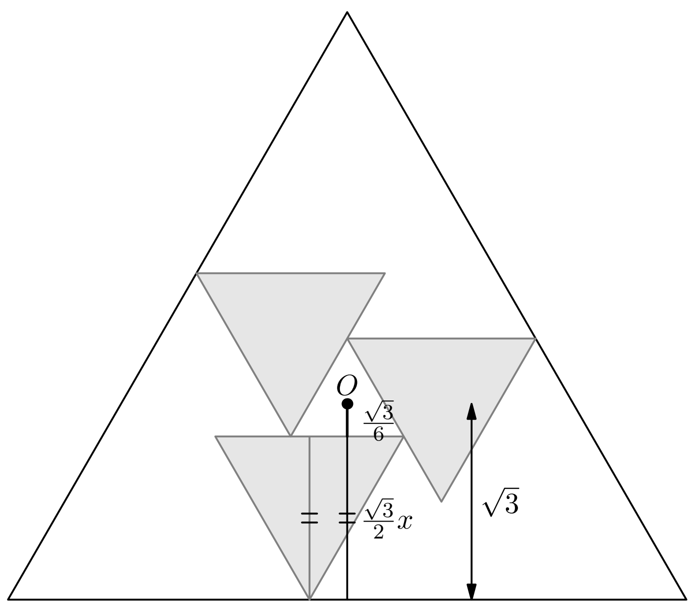


A smaller equilateral triangle formed between the three congruent equilateral triangles has side length 1. Compute x.

Proposed by: Rishabh Das

Answer: $\boxed{\frac{5}{3}}$

Solution:



Let x be the side length of the shaded triangles. Note that the centers of the triangles with side lengths 1 and 6 coincide; call this common center O.

The distance from O to a side of the equilateral triangle with side length 1 is $\sqrt{3}/6$. Similarly the distance from O to a side of the equilateral triangle with side length 6 is $\sqrt{3}$. Notice the difference of these two distances is exactly the length of the altitude of one of shaded triangles. So

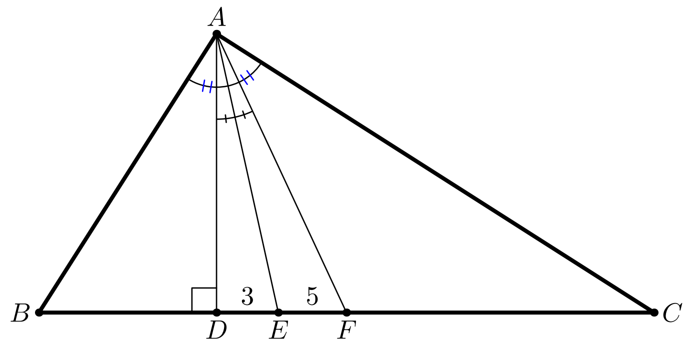
$$\sqrt{3} - \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{2}x \implies x = \boxed{\frac{5}{3}}.$$

2. Let ABC be a triangle with $\angle BAC = 90°$. Let D, E, and F be the feet of altitude, angle bisector, and median from A to BC, respectively. If $DE = 3$ and $EF = 5$, compute the length of BC.

   Proposed by: Jerry Liang

   Answer: $\boxed{20}$

   Solution 1:



   Since F is the circumcenter of $\triangle ABC$, we have that AE bisects $\angle DAF$. So by the angle bisector theorem, we can set $AD = 3x$ and $AF = 5x$. Applying Pythagorean theorem to $\triangle ADE$ then gives

$$(3x)^2 + (5+3)^2 = (5x)^2 \implies x = 2.$$

   So $AF = 5x = 10$ and $BC = 2AF = \boxed{20}$.

   Solution 2: Let $BF = FC = x$. We know that $\triangle BAD \sim \triangle ACD$ so $\frac{BA}{AC} = \frac{BD}{DA} = \frac{DA}{DC}$ and thus $\frac{BA}{AC} = \sqrt{\frac{BD}{DC}} = \sqrt{\frac{x-8}{x+8}}$. By Angle Bisector Theorem, we also have $\frac{AB}{AC} = \frac{BE}{EC} = \frac{x-5}{x+5}$, which means that

$$\sqrt{\frac{x-8}{x+8}} = \frac{x-5}{x+5} \implies (x-8)(x+5)^2 = (x+8)(x-5)^2$$

   which expands to

$$x^3 + 2x^2 - 55x - 200 = x^3 - 2x^2 - 55x + 200 \implies 4x^2 = 400.$$

   This solves to $x = 10$, and so $BC = 2x = \boxed{20}$.

3. Let Ω and ω be circles with radii 123 and 61, respectively, such that the center of Ω lies on ω. A chord of Ω is cut by ω into three segments, whose lengths are in the ratio $1 : 2 : 3$ in that order. Given that this chord is not a diameter of Ω, compute the length of this chord.

   Proposed by: Benjamin Kang, Holden Mui, Pitchayut Saengrungkongka

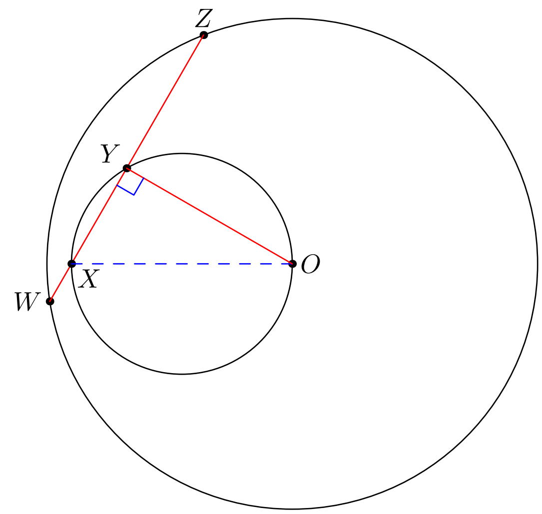
   Answer: $\boxed{42}$

   Solution: Denote the center of Ω as O. Let the chord intersect the circles at W, X, Y, Z so that $WX = t$, $XY = 2t$, and $YZ = 3t$. Notice that Y is the midpoint of WZ; hence $\overline{OY} \perp \overline{WXYZ}$.

   The fact that $\angle OYX = 90°$ means X is the antipode of O on ω, so $OX = 122$. Now applying power of point to X with respect to Ω gives

$$245 = 123^2 - OX^2 = WX \cdot XZ = 5t^2 \implies t = 7.$$

   Hence the answer is $6t = \boxed{42}$.

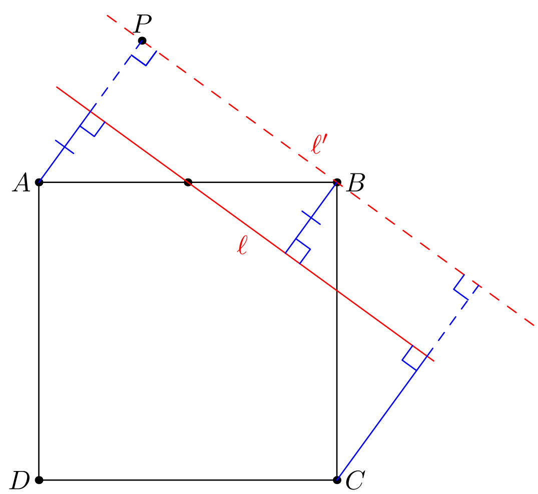


4. Let $ABCD$ be a square, and let ℓ be a line passing through the midpoint of segment $\overline{AB}$ that intersects segment $\overline{BC}$. Given that the distances from A and C to ℓ are 4 and 7, respectively, compute the area of $ABCD$.

   Proposed by: Ethan Liu

   Answer: $\boxed{185}$

   Solution:



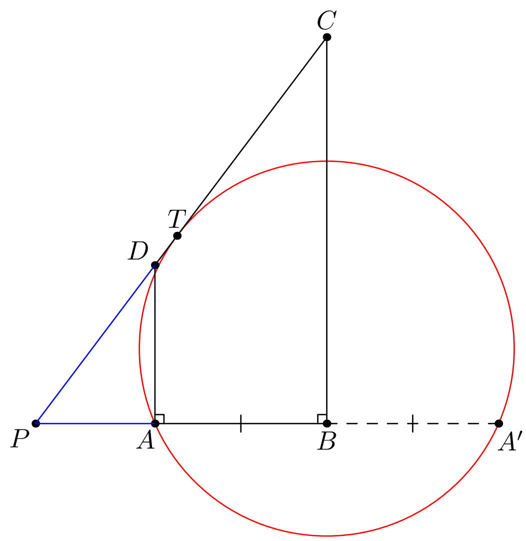
Consider the line ℓ' through B parallel to ℓ, and drop perpendiculars from A to ℓ' and C to ℓ'. Note that because ℓ passes through the midpoint of segment AB, the distance from B to ℓ is 4. Thus, the distances from A to ℓ' and from C to ℓ' are $4 + 4 = 8$ and $4 + 7 = 11$, respectively. Let P be the foot from A to ℓ'. Rotating the square 90° from B to A sends the altitude from C to ℓ' to the segment along ℓ' between B and the foot from A to ℓ'; hence $BP = 11$. So the side length of the square is $\sqrt{AP^2 + BP^2} = \sqrt{8^2 + 11^2}$, which means the area of the square is $8^2 + 11^2 = \boxed{185}$.

5. Let $ABCD$ be a convex trapezoid such that $\angle DAB = \angle ABC = 90^\circ$, $DA = 2$, $AB = 3$, and $BC = 8$. Let ω be a circle passing through A and tangent to segment $\overline{CD}$ at point T. Suppose that the center of ω lies on line BC. Compute CT.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{4\sqrt{5}-\sqrt{7}}$

Solution:



Let A' be the reflection of A across BC, and let $P = AB \cap CD$. Then since the center of ω lies on BC, we have that ω passes through A'. Thus, by power of a point, $PT^2 = PA \cdot PA'$. By similar triangles, we have

$$\frac{PA}{AD} = \frac{PB}{BC} \implies \frac{PA}{2} = \frac{PA+3}{8} \implies PA = 1,$$

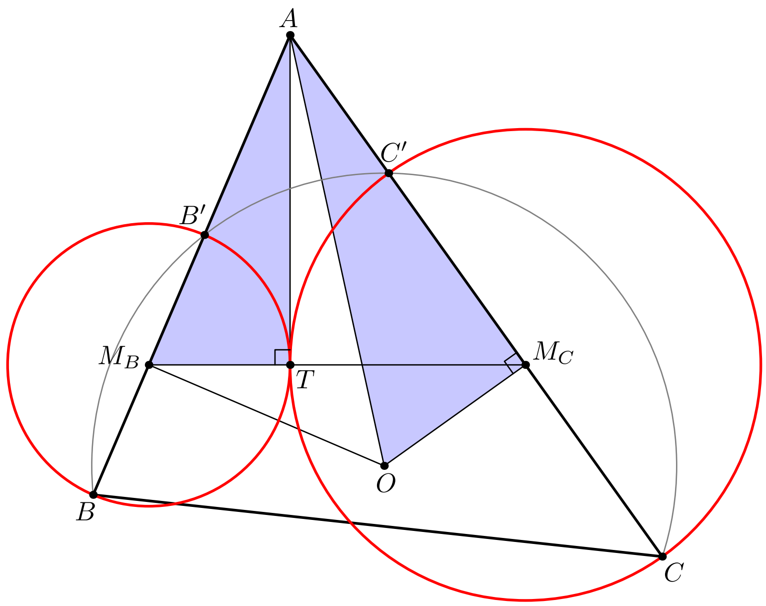
and $A'P = 1 + 2 \cdot 3 = 7$, so $PT = \sqrt{7}$. But by the Pythagorean Theorem, $PC = \sqrt{PB^2 + BC^2} = 4\sqrt{5}$, and since T lies on segment CD, it lies between C and P, so $CT = \boxed{4\sqrt{5}-\sqrt{7}}$.

6. In triangle ABC, a circle ω with center O passes through B and C and intersects segments $\overline{AB}$ and $\overline{AC}$ again at B' and C', respectively. Suppose that the circles with diameters BB' and CC' are externally tangent to each other at T. If $AB = 18$, $AC = 36$, and $AT = 12$, compute AO.

Proposed by: Ethan Liu

Answer: $\boxed{\frac{65}{3}}$

Solution 1:



By Radical Axis Theorem, we know that AT is tangent to both circles. Moreove, consider power of a point A with respect to these three circles, we have $AB \cdot AB' = AT^2 = AC \cdot AC'$. Thus $AB' = \frac{12^2}{18} = 8$, and $AC' = \frac{12^2}{36} = 4$. Consider the midpoints M_B, M_C of segments $\overline{BB'}$, $\overline{CC'}$, respectively. We have $\angle OM_BA = \angle OM_CA = 90°$, so O is the antipode of A in (AM_BM_C). Notice that $\triangle AM_BT \sim \triangle AOM_C$, so $\frac{AO}{AM_C} = \frac{AM_B}{AT}$. Now, we can do the computations as follow:

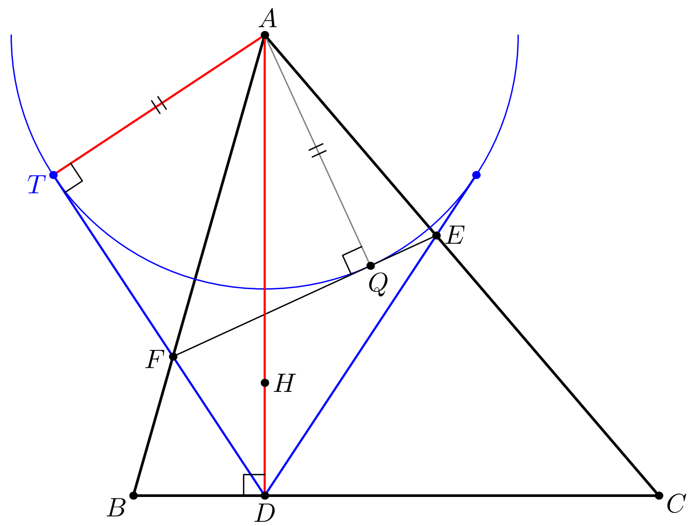
$$\begin{aligned} AO &= \frac{AM_B \cdot AM_C}{AT} \\ &= \left(\frac{AB + AB'}{2}\right)\left(\frac{AC + AC'}{2}\right)\frac{1}{AT} \\ &= \left(\frac{8+18}{2}\right)\left(\frac{36+4}{2}\right)\frac{1}{12} = \boxed{\frac{65}{3}}. \end{aligned}$$

7. Let ABC be an acute triangle. Let D, E, and F be the feet of altitudes from A, B, and C to sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, and let Q be the foot of altitude from A to line EF. Given that $AQ = 20$, $BC = 15$, and $AD = 24$, compute the perimeter of triangle DEF.

   Proposed by: Isabella Zhu

   Answer: $\boxed{8\sqrt{11}}$

   Solution:



Note that A is the excenter of $\triangle DEF$ and AQ is the length of the exradius. Let T be the tangency point of the A-excircle to line DF. We have $AQ = AT = 20$. It is well known that the length of DT is the semiperimeter of DEF. Note that $\triangle ADT$ is a right triangle, so

$$AT^2 + DT^2 = AD^2$$

which implies

$$DT = \sqrt{24^2 - 20^2} = 4\sqrt{11}.$$

Thus, the perimeter of $\triangle DEF$ is $2 \cdot 4\sqrt{11} = \boxed{8\sqrt{11}}$.

8. Let $ABTCD$ be a convex pentagon with area 22 such that $AB = CD$ and the circumcircles of triangles TAB and TCD are internally tangent. Given that $\angle ATD = 90^\circ$, $\angle BTC = 120^\circ$, $BT = 4$, and $CT = 5$, compute the area of triangle TAD.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{64(2-\sqrt{3})}$

   Solution: Paste $\triangle TCD$ outside the pentagon to get $\triangle ABX \cong \triangle DCT$. From the tangent circles condition, we get

$$\begin{aligned}\angle XBT &= 360^\circ - \angle XBA - \angle ABT \\ &= 360^\circ - \angle DCT - \angle ABT \\ &= 360^\circ - 270^\circ = 90^\circ\end{aligned}$$

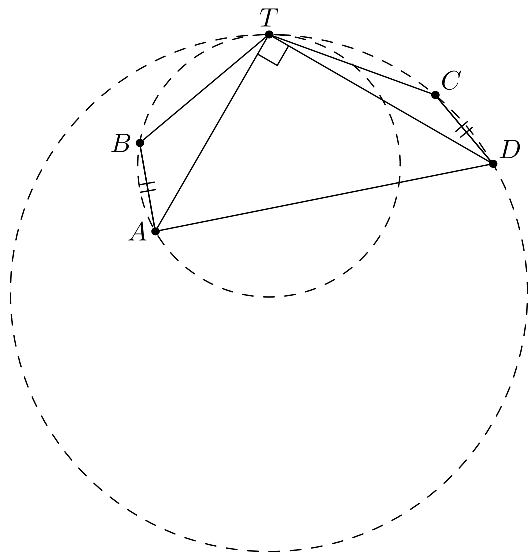
$$\begin{aligned}\angle XAT &= 90^\circ - \angle BXA - \angle ATB \\ &= 90^\circ - \angle CTD - \angle ATB \\ &= 90^\circ - (120^\circ - 90^\circ) = 60^\circ.\end{aligned}$$

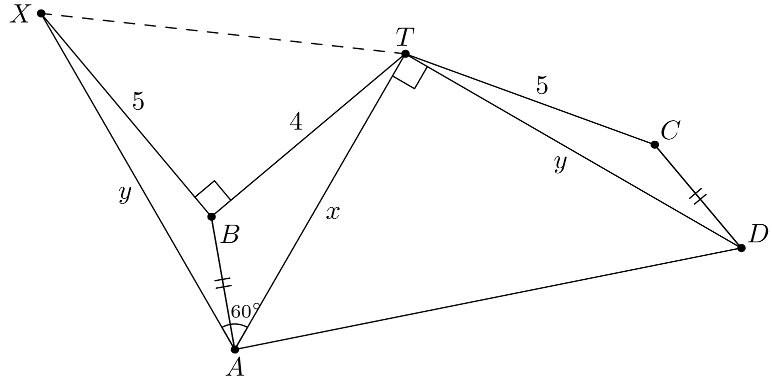
   Moreover, if $x = AT$ and $y = TD$, then notice that

$$\begin{aligned}[ABTCD] &= [ABT] + [CDT] + [ATD] \\ &= [XAT] - [XBT] + [ATD] \\ &= \frac{1}{2}xy\sin 60^\circ - \frac{1}{2}\cdot 4\cdot 5 + \frac{1}{2}xy \\ &= \frac{2+\sqrt{3}}{4}xy - 10,\end{aligned}$$

so we have

$$xy = 32 \cdot \frac{4}{2+\sqrt{3}} = 128(2-\sqrt{3}) \implies [ATD] = \frac{1}{2}xy = \boxed{64(2-\sqrt{3})}.$$



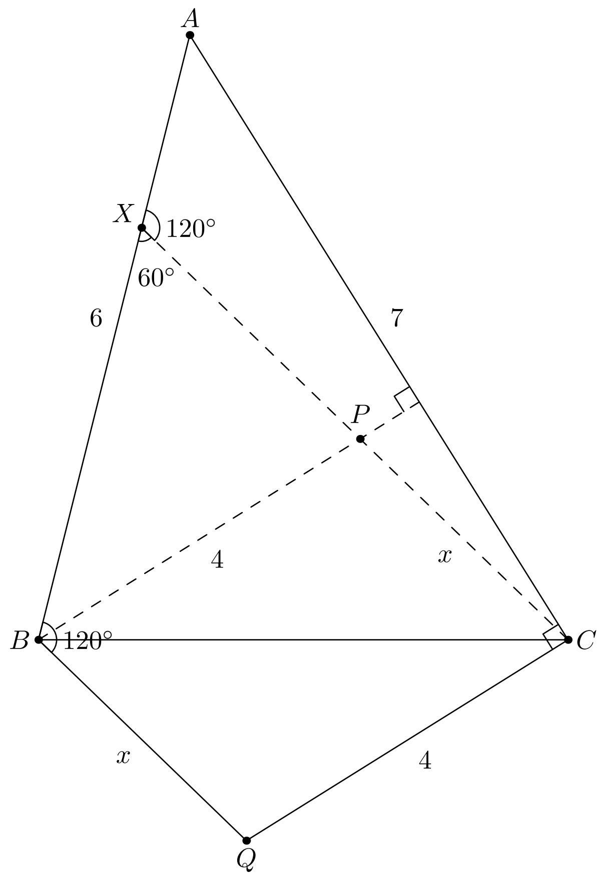


9. Let ABC be a triangle. Let X be the point on side $\overline{AB}$ such that $\angle BXC = 60^\circ$. Let P be the point on segment $\overline{CX}$ such that $BP \perp AC$. Given that $AB = 6$, $AC = 7$, and $BP = 4$, compute CP.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{\sqrt{38}-3}$

   Solution: Construct parallelogram $BPCQ$. We have $CQ = 4$, $\angle ACQ = 90^\circ$, and $\angle ABQ = 120^\circ$. Thus, $AQ = \sqrt{AC^2 + CQ^2} = \sqrt{65}$, so if $x = CP = BQ$, then by Law of Cosine, $x^2 + 6x + 6^2 = 65$. Solving this gives the answer $x = \boxed{\sqrt{38}-3}$.



10. Suppose point P is inside quadrilateral $ABCD$ such that

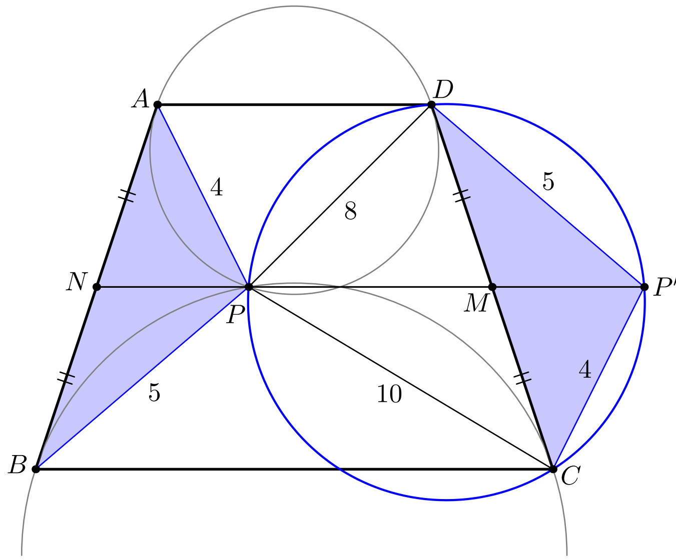
$$\angle PAB = \angle PDA,$$
$$\angle PAD = \angle PDC,$$
$$\angle PBA = \angle PCB, \text{ and}$$
$$\angle PBC = \angle PCD.$$

If $PA = 4$, $PB = 5$, and $PC = 10$, compute the perimeter of $ABCD$.

Proposed by: Rishabh Das

Answer: $\boxed{\frac{9\sqrt{410}}{5}}$

Solution:



First of all, note that the angle conditions imply that $\angle BAD + \angle ABC = 180^\circ$, so the quadrilateral is a trapezoid with $AD \parallel BC$. Moreover, they imply AB and CD are both tangent to (PAD) and (PBC); in particular $AB = CD$ or $ABCD$ is isosceles trapezoid. Since the midpoints of AD and BC clearly lie on the radical axis of the two circles, P is on the midline of the trapezoid.

Reflect $\triangle PAB$ over the midline and translate it so that $D = B'$ and $C = A'$. Note that P' is still on the midline. The angle conditions now imply $PDP'C$ is cyclic, and PP' bisects CD. This means $10 \cdot 4 = PC \cdot CP' = PD \cdot DP' = 5 \cdot PD$, so $PD = 8$.

Now $PDP'C$ is a cyclic quadrilateral with side lengths $10, 8, 5, 4$ in that order. Using standard cyclic quadrilateral facts (either law of cosines or three applications on Ptolemy on the three possible quadrilaterals formed with these side lengths) we get $CD = \frac{2\sqrt{410}}{5}$ and $PP' = \frac{\sqrt{410}}{2}$. Finally, note that PP' is equal to the midline of the trapezoid, so the final answer is

$$2 \cdot CD + 2 \cdot PP' = \boxed{\frac{9\sqrt{410}}{5}}.$$