HMMT February 2024

February 17, 2024

Guts Round

1. **[5]** Compute the sum of all integers n such that $n^2 - 3000$ is a perfect square.

   Proposed by: Holden Mui

   Answer: $\boxed{0}$

   Solution: If $n^2 - 3000$ is a square, then $(-n)^2 - 3000$ is also a square, so the sum is $\boxed{0}$.

2. **[5]** Jerry and Neil have a 3-sided die that rolls the numbers 1, 2, and 3, each with probability $\frac{1}{3}$. Jerry rolls first, then Neil rolls the die repeatedly until his number is at least as large as Jerry's. Compute the probability that Neil's final number is 3.

   Proposed by: Rishabh Das

   Answer: $\boxed{\frac{11}{18}}$

   Solution: If Jerry rolls k, then there is a $\frac{1}{4-k}$ probability that Neil's number is 3, since Neil has an equal chance of rolling any of the $4 - k$ integers not less than k. Thus, the answer is

$$\frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \boxed{\frac{11}{18}}.$$

3. **[5]** Compute the number of even positive integers $n \leq 2024$ such that $1, 2, \ldots, n$ can be split into $\frac{n}{2}$ pairs, and the sum of the numbers in each pair is a multiple of 3.

   Proposed by: Rishabh Das

   Answer: $\boxed{675}$

   Solution: There have to be an even number of multiples of 3 at most n, so this means that $n \equiv 0, 2 \pmod{6}$. (We can also say that there should be an equal number of $1 \pmod{3}$ and $2 \pmod{3}$ numbers, which gives the same restriction.)

   We claim that all these work. We know there are an even number of multiples of 3, so we can pair them; then we can pair $3k + 1$ and $3k + 2$ for all k.

   This means the answer is $\frac{2022}{3} + 1 = \boxed{675}$.

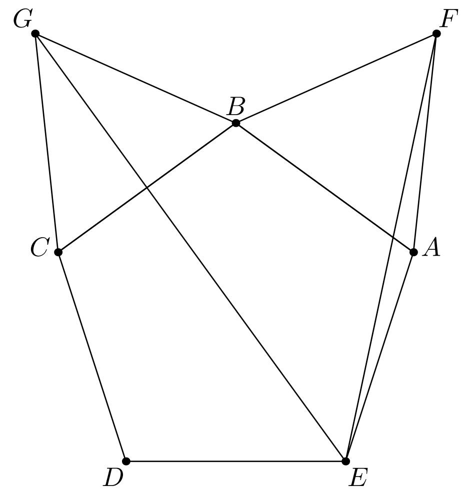
4. **[5]** Equilateral triangles ABF and BCG are constructed outside regular pentagon $ABCDE$. Compute $\angle FEG$.

   Proposed by: Karthik Venkata Vedula

   Answer: $\boxed{48^\circ = \frac{4\pi}{15}}$

   Solution: We have $\angle FEG = \angle AEG - \angle AEF$. Since EG bisects $\angle AED$, we get $\angle AEG = 54^\circ$.

   Now, $\angle EAF = 108^\circ + 60^\circ = 168^\circ$. Since triangle EAF is isosceles, this means $\angle AEF = 6^\circ$, so the answer is $54^\circ - 6^\circ = \boxed{48^\circ}$.



5. **[6]** Let a, b, and c be real numbers such that

$$\begin{aligned} a+b+c &= 100, \\ ab+bc+ca &= 20, \text{ and} \\ (a+b)(a+c) &= 24. \end{aligned}$$

Compute all possible values of bc.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{224, -176}$

Solution: We first expand the left-hand-side of the third equation to get $(a+b)(a+c) = a^2 + ac + ab + bc = 24$. From this, we subtract the second equation to obtain $a^2 = 4$, so $a = \pm 2$.

If $a = 2$, plugging into the first equation gives us $b + c = 98$ and plugging into the second equation gives us $2(b+c) + bc = 20 \Rightarrow 2(98) + bc = 20 \Rightarrow bc = -176$.

Then, if $a = -2$, plugging into the first equation gives us $b + c = 102$, and plugging into the second equation gives us $-2(b+c) + bc = 20 \Rightarrow -2(102) + bc = 20 \Rightarrow bc = 224$.

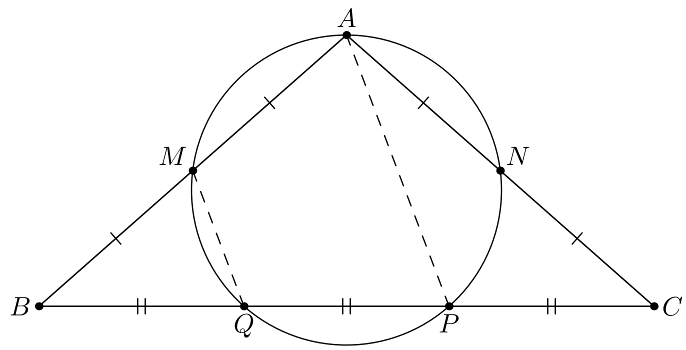
Therefore, the possible values of bc are $\boxed{224, -176}$.

6. **[6]** In triangle ABC, points M and N are the midpoints of AB and AC, respectively, and points P and Q trisect BC. Given that A, M, N, P, and Q lie on a circle and $BC = 1$, compute the area of triangle ABC.

Proposed by: Rishabh Das

Answer: $\boxed{\frac{\sqrt{7}}{12}}$

Solution: Note that $MP \parallel AQ$, so $AMPQ$ is an isosceles trapezoid. In particular, we have $AM = MB = BP = PQ = \frac{1}{3}$, so $AB = \frac{2}{3}$. Thus ABC is isoceles with base 1 and legs $\frac{2}{3}$, and the height from A to BC is $\frac{\sqrt{7}}{6}$, so the area is $\boxed{\frac{\sqrt{7}}{12}}$.



7. **[6]** Positive integers a, b, and c have the property that a^b, b^c, and c^a end in 4, 2, and 9, respectively. Compute the minimum possible value of $a + b + c$.

   Proposed by: Derek Liu

   Answer: $\boxed{17}$

   Solution: This minimum is attained when $(a, b, c) = (2, 2, 13)$. To show that we cannot do better, observe that a must be even, so c ends in 3 or 7. If $c \geq 13$, since a and b are even, it's clear $(2, 2, 13)$ is optimal. Otherwise, $c = 3$ or $c = 7$, in which case b^c can end in 2 only when b ends in 8. However, no eighth power ends in 4, so we would need $b \geq 18$ (and $a \geq 2$), which makes the sum $2 + 18 + 3 = 23$ larger than $\boxed{17}$.

8. **[6]** Three points, A, B, and C, are selected independently and uniformly at random from the interior of a unit square. Compute the expected value of $\angle ABC$.

   Proposed by: Akash Das

   Answer: $\boxed{60^\circ = \frac{\pi}{3}}$

   Solution: Since $\angle ABC + \angle BCA + \angle CAB = 180^\circ$ for all choices of A, B, and C, the expected value is $\boxed{60^\circ}$.

9. **[7]** Compute the sum of all positive integers n such that $n^2 - 3000$ is a perfect square.

   Proposed by: Holden Mui, Pitchayut Saengrungkongka, Rishabh Das

   Answer: $\boxed{1872}$

   Solution: Suppose $n^2 - 3000 = x^2$, so $n^2 - x^2 = 3000$. This factors as $(n - x)(n + x) = 3000$. Thus, we have $n - x = 2a$ and $n + x = 2b$ for some positive integers a, b such that $ab = 750$ and $a < b$. Therefore, we have $n = a + b$, so the sum will be just sum of divisors of $750 = 2 \cdot 3 \cdot 5^3$, which is

   $$(1 + 2)(1 + 3)(1 + 5 + 25 + 125) = \boxed{1872}.$$

   Remark. Problem 1 and 9 have slightly different statements.

   1. Compute the sum of all integers n such that $n^2 - 3000$ is a perfect square.
   9. Compute the sum of all **positive** integers n such that $n^2 - 3000$ is a perfect square.

   There are 86 teams participating in the Guts rounds. Of these, the distribution of answers to Problem 1 is as follows:

- 49 teams submitted 0, the correct answer.
- 17 teams submitted 1872, the correct answer to Problem 9.
- 3 teams submitted 3744, twice the correct answer to Problem 9.
- 2 teams submitted each of 55 and 744.
- 1 team submitted each of the following answers: 20, 548, 1404, 1586, 1772, 1807, 1817, 1882, 2184, 2746, 4680, 7488, and 9360.

The distribution of answers to Problem 9 is as follows:

- 69 teams submitted 1872, the correct answer.
- 5 teams submitted 3744, twice the correct answer.
- 2 teams submitted 2184.
- 1 team submitted each of the following answers: 55, 205, 548, 1560, 1764, 1772, 1832, 1867, 1893, and 3634.

10. **[7]** Alice, Bob, and Charlie are playing a game with 6 cards numbered 1 through 6. Each player is dealt 2 cards uniformly at random. On each player's turn, they play one of their cards, and the winner is the person who plays the median of the three cards played. Charlie goes last, so Alice and Bob decide to tell their cards to each other, trying to prevent him from winning whenever possible. Compute the probability that Charlie wins regardless.

Proposed by: Ethan Liu

Answer: $\boxed{\frac{2}{15}}$

Solution: If Alice has a card that is adjacent to one of Bob's, then Alice and Bob will play those cards as one of them is guaranteed to win. If Alice and Bob do not have any adjacent cards, since Charlie goes last, Charlie can always choose a card that will win.

Let A denote a card that is held by Alice and B denote a card that is held by Bob. We will consider the ascneding order of which Alice and Bob's cards are held.

If the ascending order in which Alice and Bob's cards are held are $ABAB$ or $BABA$, then Charlie cannot win. In these 2 cases, there will always be 2 consecutive cards where one is held by Alice and the other is held by Bob. Therefore, the only cases we need to consider are the ascending orders $AABB$, $ABBA$, and their symmetric cases.

In the case $AABB$, we must make sure that the larger card Alice holds and the smaller card Bob holds are not consecutive. Alice can thus have $\{1,2\}, \{2,3\}$, or $\{1,3\}$. Casework on what Bob can have yields 5 different combinations of pairs of cards Alice and Bob can hold. Since this applies to the symmetric case $BBAA$ as well, we get 10 different combinations.

In the case $ABBA$, we see that Alice's cards must be $\{1,6\}$ and Bob's cards must be $\{3,4\}$. Considering the symmetric case $BAAB$ as well, this gives us 2 more combinations.

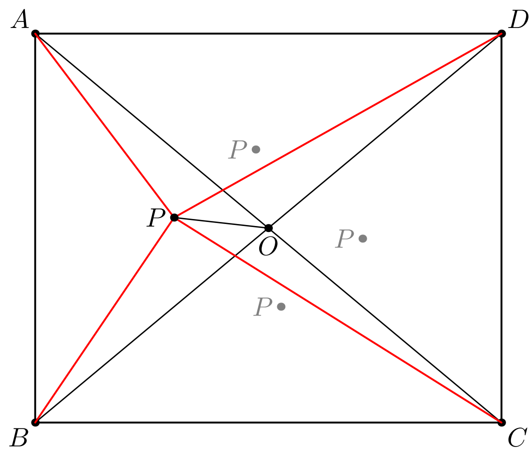
Thus, there are 12 total possible combinations of Alice's and Bob's cards such that Charlie will win regardless. The total number of ways to choose Alice's and Bob's cards is given by $\binom{6}{2}\binom{4}{2} = 90$, so the probability that Charlie is guaranteed to win is $\frac{12}{90} = \boxed{\frac{2}{15}}$.

11. **[7]** Let $ABCD$ be a rectangle such that $AB = 20$ and $AD = 24$. Point P lies inside $ABCD$ such that triangles PAC and PBD have areas 20 and 24, respectively. Compute all possible areas of triangle PAB.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{98,\ 118,\ 122,\ 142}$

Solution:



There are four possible locations of P as shown in the diagram. Let O be the center. Then, $[PAO] = 10$ and $[PBO] = 12$. Thus, $[PAB] = [AOB] \pm [PAO] \pm [PBO] = 120 \pm 10 \pm 12$, giving the four values $\boxed{98,\ 118,\ 122,\text{ and }142}$.

12. **[7]** Compute the number of quadruples (a, b, c, d) of positive integers satisfying

$$12a + 21b + 28c + 84d = 2024.$$

Proposed by: Rishabh Das

Answer: $\boxed{2024}$

Solution: Looking at the equation mod 7 gives $a \equiv 3 \pmod 7$, so let $a = 7a' + 3$. Then mod 4 gives $b \equiv 0 \pmod 4$, so let $b = 4b'$. Finally, mod 3 gives $c \equiv 2 \pmod 3$, so let $c = 3c' + 2$.

Now our equation yields

$$84a' + 84b' + 84c' + 84d = 2024 - 3 \cdot 12 - 2 \cdot 28 = 1932 \implies a' + b' + c' + d = 23.$$

Since a, b, c, d are positive integers, we have a' and c' are nonnegative and b' and d are positive. Thus, let $b'' = b' + 1$ and $d' = d + 1$, so a', b'', c', d' are nonnegative integers summing to 21. By stars and bars, there are $\binom{24}{3} = \boxed{2024}$ such solutions.

13. **[9]** Mark has a cursed six-sided die that never rolls the same number twice in a row, and all other outcomes are equally likely. Compute the expected number of rolls it takes for Mark to roll every number at least once.

Proposed by: Albert Wang

Answer: $\boxed{\frac{149}{12}}$

Solution: Suppose Mark has already rolled n unique numbers, where $1 \leq n \leq 5$. On the next roll, there are 5 possible numbers he could get, with $6 - n$ of them being new. Therefore, the probability of getting another unique number is $\frac{6-n}{5}$, so the expected number of rolls before getting another unique number is $\frac{5}{6-n}$. Since it always takes 1 roll to get the first number, the expected total number of rolls is $1 + \frac{5}{5} + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + \frac{5}{1} = \boxed{\frac{149}{12}}$.

14. **[9]** Compute the smallest positive integer such that, no matter how you rearrange its digits (in base ten), the resulting number is a multiple of 63.

Proposed by: Arul Kolla

Answer: $\boxed{111\,888}$

Solution: First, the number must be a multiple of 9 and 7. The first is easy to check and holds for all permutations. Note that when two adjacent digits a and b are swapped, the number changes by $9(a-b) \cdot 10^k$ (we disregard sign), so $9(a-b)$ must also be a multiple of 63 for all digits a and b. In particular, this is sufficient, since a permutation can be represented as a series of transpositions.

This means that $a - b$ must be a multiple of 7 for all digits a and b, so either all digits are equal or they are in $\{0, 7\}$, $\{1, 8\}$, or $\{2, 9\}$. We find the minimum for each case separately.

We first provide the following useful fact: the first repunit (numbers 1, 11, 111, ...) that is a multiple of 7 is 111111. This is because $10 \bmod 7 = 3$, and 3 is a generator modulo 7 (of course, you can just compute the powers of 3 by hand, and it will not take much longer).

If a number $k \cdot 1 \ldots 1$ is a multiple of 63, then either k or $1 \ldots 1$ is a multiple of 7; if it is k, then it's clear that we need $777\,777\,777$ to make the sum a multiple of 9. If $1 \ldots 1$ is a multiple of 7, then it is at least $111\,111$, then to make a multiple of 9, we need $333\,333$.

If the only digits are 7 and 0, then we need at least nine sevens to make the digit sum a multiple of nine, which has more digits than $333\,333$.

If the only digits are 8 and 1, then we can note that since 8 and 1 are both $1 \pmod 7$, these numbers are equivalent to the repunits modulo 7, so such numbers have at least six digits. The best such six-digit number with digits summing to a multiple of 9 is $111\,888$, which is our new candidate.

If the only digits are 9 and 2, then by analogous logic such numbers have at least six digits. But the smallest such number is $999\,999$, which is not better.

So our best answer is $\boxed{111\,888}$. It works.

15. **[9]** Let $a \star b = ab - 2$. Compute the remainder when $(((579 \star 569) \star 559) \star \cdots \star 19) \star 9$ is divided by 100.

Proposed by: Rishabh Das

Answer: $\boxed{29}$

Solution: Note that

$$(10a + 9) \star (10b + 9) = (100ab + 90a + 90b + 81) - 2 \equiv 90(a + b) + 79 \pmod{100},$$

so throughout our process all numbers will end in 9, so we will just track the tens digit. Then the "new operation" is

$$a \dagger b \equiv -(a + b) + 7 \mod 10,$$

where a and b track the tens digits. Now

$$(a \dagger b) \dagger c \equiv (-(a + b) + 7) \dagger c \equiv a + b - c \mod 10.$$

Thus, our expression has tens digit congruent to

$$-0 + 1 - 2 + 3 - \cdots - 54 + 55 - 56 - 57 + 7 \equiv -28 - 57 + 7 \equiv 2 \mod 10,$$

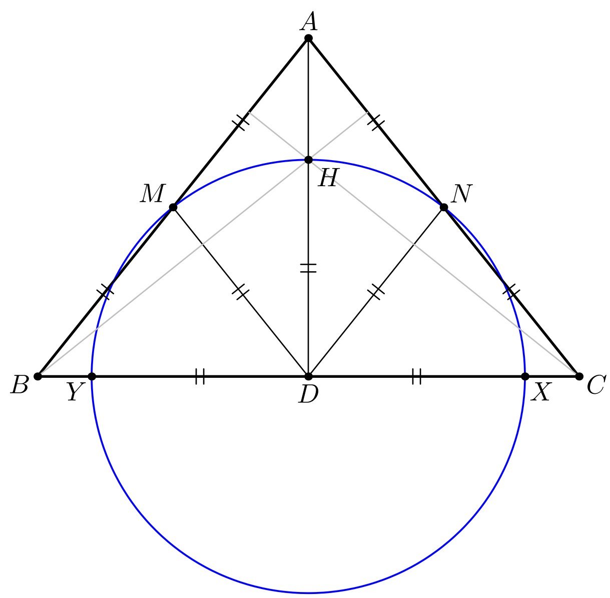
making the answer $\boxed{29}$.

16. **[9]** Let ABC be an acute isosceles triangle with orthocenter H. Let M and N be the midpoints of sides $\overline{AB}$ and $\overline{AC}$, respectively. The circumcircle of triangle MHN intersects line BC at two points X and Y. Given $XY = AB = AC = 2$, compute BC^2.

Proposed by: Andrew Wen

Answer: $\boxed{2(\sqrt{17}-1)}$

Solution:



Let D be the foot from A to BC, also the midpoint of BC. Note that $DX = DY = MA = MB = MD = NA = NC = ND = 1$. Thus, $MNXY$ is cyclic with circumcenter D and circumradius 1. H lies on this circle too, hence $DH = 1$.

If we let $DB = DC = x$, then since $\triangle HBD \sim \triangle BDA$,

$$BD^2 = HD \cdot AD \implies x^2 = \sqrt{4-x^2} \implies x^4 = 4 - x^2 \implies x^2 = \frac{\sqrt{17}-1}{2}.$$

Our answer is $BC^2 = (2x)^2 = 4x^2 = \boxed{2(\sqrt{17}-1)}$

17. **[11]** The numbers $1, 2, \ldots, 20$ are put into a hat. Claire draws two numbers from the hat uniformly at random, $a < b$, and then puts them back into the hat. Then, William draws two numbers from the hat uniformly at random, $c < d$.

Let N denote the number of integers n that satisfy exactly one of $a \le n \le b$ and $c \le n \le d$. Compute the probability N is even.

Proposed by: Rishabh Das

Answer: $\boxed{\frac{181}{361}}$

Solution: The number of integers that satisfy exactly one of the two inequalities is equal to the number of integers that satisfy the first one, plus the number of integers that satisfy the second one, minus twice the number of integers that satisfy both. Parity-wise, this is just the number of integers that satisfy the first one, plus the number of integers that satisfy the second one.

The number of integers that satisfy the first one is $b-a+1$. The probability this is even is $\frac{10}{19}$, and odd is $\frac{9}{19}$. This means the answer is

$$\frac{10^2+9^2}{19^2} = \boxed{\frac{181}{361}}.$$

18. **[11]** An ordered pair (a, b) of positive integers is called *spicy* if $\gcd(a+b, ab+1) = 1$. Compute the probability that both $(99, n)$ and $(101, n)$ are spicy when n is chosen from $\{1, 2, \ldots, 2024!\}$ uniformly at random.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{96}{595}}$

Solution: We claim that (a, b) is spicy if and only if both $\gcd(a+1, b-1) = 1$ and $\gcd(a-1, b+1) = 1$.

To prove the claim, we note that

$$\gcd(a+b, ab+1) = \gcd(a+b, b(-b)+1) = \gcd(a+b, b^2-1).$$

Hence, we have

$$\begin{aligned} \gcd(a+b, ab+1) = 1 &\iff \gcd(a+b, b^2-1) = 1 \\ &\iff \gcd(a+b, b-1) = 1 \text{ and } \gcd(a+b, b+1) = 1 \\ &\iff \gcd(a+1, b-1) = 1 \text{ and } \gcd(a-1, b+1) = 1, \end{aligned}$$

proving the claim.

Thus, n works if and only if all following four conditions hold:

- $\gcd(n+1, 98) = 1$, or equivalently, n is neither $-1 \pmod 2$ nor $-1 \pmod 7$;
- $\gcd(n-1, 100) = 1$, or equivalently, n is neither $1 \pmod 2$ nor $1 \pmod 5$;
- $\gcd(n+1, 100) = 1$, or equivalently, n is neither $-1 \pmod 2$ nor $-1 \pmod 5$; and
- $\gcd(n-1, 102) = 1$, or equivalently, n is neither $1 \pmod 2$, $1 \pmod 3$, nor $1 \pmod{17}$.

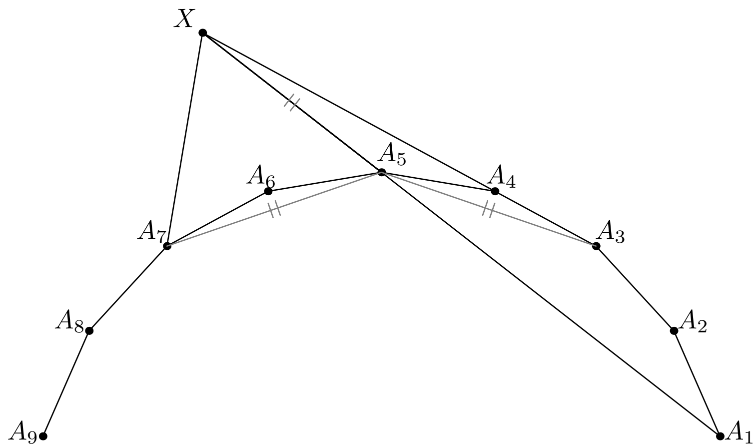
Thus, there are 1, 2, 3, 6, 17 possible residues modulo 2, 3, 5, 7, and 17, respectively. The residues are uniformly distributed within $\{1, 2, \ldots, 2024!\}$. Hence, the answer is $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{6}{7} \cdot \frac{16}{17} = \boxed{\frac{96}{595}}$.

19. **[11]** Let $A_1A_2 \ldots A_{19}$ be a regular nonadecagon. Lines A_1A_5 and A_3A_4 meet at X. Compute $\angle A_7XA_5$.

Proposed by: Nithid Anchaleenukoon

Answer: $\boxed{\frac{1170}{19}^\circ = \frac{13\pi}{38}}$

Solution:



Inscribing the nondecagon in a circle, note that

$$\angle A_3XA_5 = \frac{1}{2}(\widehat{A_1A_3} - \widehat{A_4A_5}) = \frac{1}{2}\widehat{A_5A_3A_4} = \angle A_5A_3X.$$

Thus $A_5X = A_5A_3 = A_5A_7$, so

$$\begin{aligned}\angle A_7XA_5 &= 90^\circ - \frac{1}{2}\angle XA_5A_7 = \frac{1}{2}\angle A_1A_5A_7 \\ &= \frac{1}{4}\widehat{A_1A_8A_7} = \frac{1}{4}\cdot\frac{13}{19}\cdot 360^\circ = \boxed{\frac{1170}{19}^\circ}.\end{aligned}$$

20. **[11]** Compute $\sqrt[4]{5508^3 + 5625^3 + 5742^3}$, given that it is an integer.

Proposed by: Rishabh Das

Answer: $\boxed{855}$

Solution: Let $a = 5625 = 75^2$ and $b = 117$. Then we have

$$5508^3 + 5265^3 + 5742^3 = (a-b)^3 + a^3 + (a+b)^3 = 3a^3 + 6ab^2 = 3a(a^2 + 2b^2).$$

We have $3a = 3^3 \cdot 5^4$, so $a^2 + 2b^2 = 3^4 \cdot (625^2 + 2 \cdot 19^2)$ should be 3 times a fourth power. This means

$$625^2 + 2 \cdot 19^2 = 3x^4$$

for some integer x. By parity, x must be odd, and also $x^2\sqrt{3} \approx 625$. Approximating $\sqrt{3}$ even as 2, we get x should be around 19. Then $x = 17$ is clearly too small, and $x = 21$ is too big. (You can also check mod 7 for this latter one.) Thus, $x = 19$. The final answer is then

$$3^2 \cdot 5 \cdot 19 = \boxed{855}.$$

21. **[12]** Kelvin the frog currently sits at $(0,0)$ in the coordinate plane. If Kelvin is at (x,y), either he can walk to any of $(x,y+1)$, $(x+1,y)$, or $(x+1,y+1)$, or he can jump to any of $(x,y+2)$, $(x+2,y)$ or $(x+1,y+1)$. Walking and jumping from (x,y) to $(x+1,y+1)$ are considered distinct actions. Compute the number of ways Kelvin can reach $(6,8)$.

Proposed by: Derek Liu

Answer: $\boxed{1831830 = 610 \cdot \binom{14}{6}}$

Solution: Observe there are $\binom{14}{6} = 3003$ up-right paths from $(0,0)$ to $(6,8)$, each of which are 14 steps long. Any two of these steps can be combined into one: UU, RR, and RU as jumps, and UR as walking from (x,y) to $(x+1,y+1)$. The number of ways to combine steps is the number of ways to group 14 actions into singles and consecutive pairs, which is $F_{15} = 610$. Every path Kelvin can take can be represented this way, so the answer is $610 \cdot 3003 = \boxed{1831830}$.

22. **[12]** Let $x < y$ be positive real numbers such that

$$\sqrt{x} + \sqrt{y} = 4 \quad \text{and} \quad \sqrt{x+2} + \sqrt{y+2} = 5.$$

Compute x.

Proposed by: Ethan Liu

Answer: $\boxed{\frac{49}{36}}$

Solution: Adding and subtracting both equations gives

$$\sqrt{x+2} + \sqrt{x} + \sqrt{y+2} + \sqrt{y} = 9$$

$$\sqrt{x+2} - \sqrt{x} + \sqrt{y+2} - \sqrt{y} = 1$$

Substitute $a = \sqrt{x} + \sqrt{x+2}$ and $b = \sqrt{y} + \sqrt{y+2}$. Then since $(\sqrt{x+2} + \sqrt{x})(\sqrt{x+2} - \sqrt{x}) = 2$, we have

$$a + b = 9$$
$$\frac{2}{a} + \frac{2}{b} = 1$$

Dividing the first equation by the second one gives

$$ab = 18, a = 3, b = 6$$

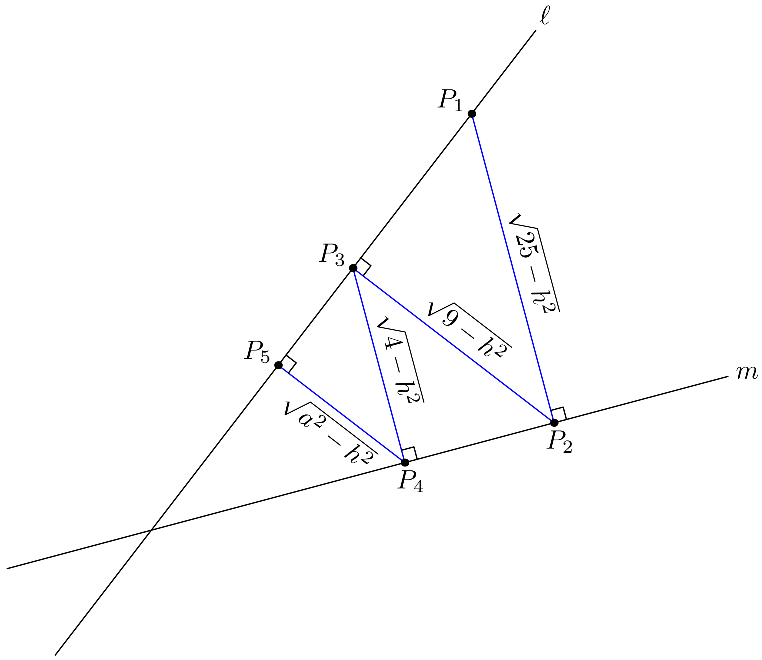
Lastly, $\sqrt{x} = \frac{\sqrt{x+2}+\sqrt{x}-(\sqrt{x+2}-\sqrt{x})}{2} = \frac{3-\frac{2}{3}}{2} = \frac{7}{6}$, so $x = \boxed{\frac{49}{36}}$.

23. **[12]** Let ℓ and m be two non-coplanar lines in space, and let P_1 be a point on ℓ. Let P_2 be the point on m closest to P_1, P_3 be the point on ℓ closest to P_2, P_4 be the point on m closest to P_3, and P_5 be the point on ℓ closest to P_4. Given that $P_1P_2 = 5$, $P_2P_3 = 3$, and $P_3P_4 = 2$, compute P_4P_5.

Proposed by: Luke Robitaille

Answer: $\boxed{\frac{\sqrt{39}}{4}}$

Solution: The figure below shows the situation of the problem when projected appropriately, which will be explained later.



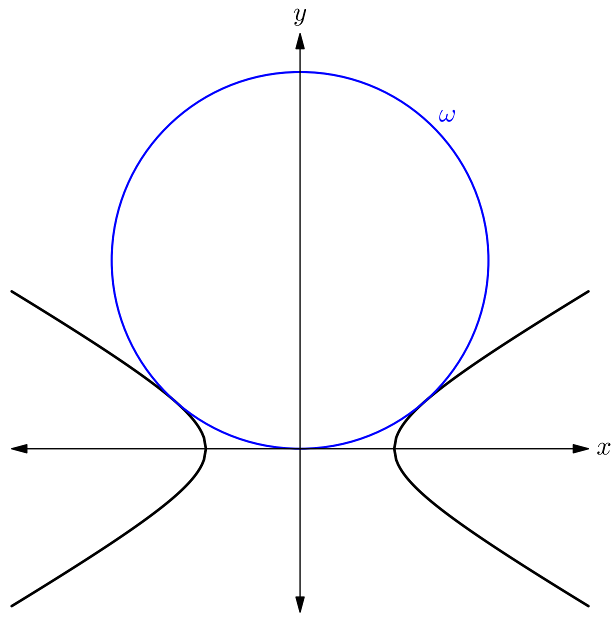
Let a be the answer. By taking the z-axis to be the cross product of these two lines, we can let the lines be on the planes $z = 0$ and $z = h$, respectively. Then, by projecting onto the xy-plane, we get the above diagram. The projected lengths of the first four segments are $\sqrt{25-h^2}$, $\sqrt{9-h^2}$, and $\sqrt{4-h^2}$, and $\sqrt{a^2-h^2}$. By similar triangles, these lengths must form a geometric progression. Therefore, $25-h^2$, $9-h^2$, $4-h^2$, a^2-h^2 is a geometric progression. By taking consecutive differences, 16, 5, $4-a^2$ is a geometric progression. Hence, $4-a^2 = \frac{25}{16} \implies a = \boxed{\frac{\sqrt{39}}{4}}$.

24. **[12]** A circle is tangent to both branches of the hyperbola $x^2 - 20y^2 = 24$ as well as the x-axis. Compute the area of this circle.

Proposed by: Karthik Venkata Vedula

Answer: $\boxed{504\pi}$

Solution 1:



Invert about the unit circle centered at the origin. ω turns into a horizontal line, and the hyperbola turns into the following:

$$\begin{aligned}\frac{x^2}{(x^2+y^2)^2} - \frac{20y^2}{(x^2+y^2)^2} = 24 &\implies x^2 - 20y^2 = 24(x^2+y^2)^2. \\ &\implies 24x^4 + (48y^2-1)x^2 + 24y^4 + 20y^2 = 0 \\ &\implies (48y^2-1)^2 \geq 4(24)(24y^4+20y^2) \\ &\implies 1 - 96y^2 \geq 1920y^2 \\ &\implies y \leq \sqrt{1/2016}.\end{aligned}$$

This means that the horizontal line in question is $y = \sqrt{1/2016}$. This means that the diameter of the circle is the reciprocal of the distance between the point and line, which is $\sqrt{2016}$, so the radius is $\sqrt{504}$, and the answer is $\boxed{504\pi}$.

Solution 2: Let a be the y-coordinate of both tangency points to the hyperbola. Then, the equation of the circle must be in the form

$$x^2 - 20y^2 + c(y-a)^2 = 24.$$

Comparing the y^2-coefficient, we see that $c = 21$. Moreover, we need it to pass through $(0,0)$, so $21a^2 = 24$. Thus, the equation of the circle is

$$x^2 + y^2 - 42ay + 21a^2 = 24 \implies x^2 + (y-21a)^2 = (21a)^2,$$

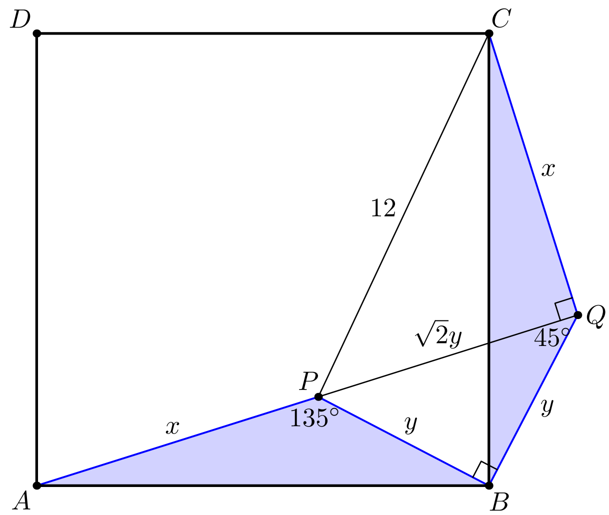
so the radius is $21a$, and the area is $(441a^2)\pi = \boxed{504\pi}$.

25. **[14]** Point P is inside a square $ABCD$ such that $\angle APB = 135^\circ$, $PC = 12$, and $PD = 15$. Compute the area of this square.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{123 + 6\sqrt{119}}$

Solution:



Let $x = AP$ and $y = BP$. Rotate $\triangle BAP$ by $90°$ around B to get $\triangle BCQ$. Then, $\triangle BPQ$ is right-isosceles, and from $\angle BQC = 135°$, we get $\angle PQC = 90°$. Therefore, by Pythagorean's theorem, $PC^2 = x^2 + 2y^2$. Similarly, $PD^2 = y^2 + 2x^2$.

Thus, $y^2 = \frac{2PC^2 - PD^2}{3} = 21$, and similarly $x^2 = 102 \implies xy = 3\sqrt{238}$.

Thus, by the Law of Cosines, the area of the square is

$$\begin{aligned} AB^2 &= AP^2 + BP^2 - 2\cos(135°)(AP)(BP) \\ &= x^2 + y^2 + \sqrt{2}xy \\ &= \boxed{123 + 6\sqrt{119}}. \end{aligned}$$

26. **[14]** It can be shown that there exists a unique polynomial P in two variables such that for all positive integers m and n,

$$P(m,n) = \sum_{i=1}^{m}\sum_{j=1}^{n}(i+j)^7.$$

Compute $P(3,-3)$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{-2445}$

Solution: Note that for integers $m > 0, n > 1$,

$$P(m,n) - P(m,n-1) = \sum_{i=1}^{m}(i+n)^7.$$

For any given positive integer m, both sides are a polynomial in n, so they must be equal as polynomials. In particular,

$$P(3,x) - P(3,x-1) = \sum_{i=1}^{3}(i+x)^7 = (x+1)^7 + (x+2)^7 + (x+3)^7$$

for all real x. Moreover, $P(3,1) - P(3,0) = P(3,1) \implies P(3,0) = 0$. Then

$$\begin{aligned} P(3,-3) &= P(3,0) - (1^7 + 2^7 + 3^7) - (0^7 + 1^7 + 2^7) - ((-1)^7 + 0^7 + 1^7) \\ &= -3^7 - 2 \cdot 2^7 - 2 = \boxed{-2445}. \end{aligned}$$

27. **[14]** A deck of 100 cards is labeled $1, 2, \ldots, 100$ from top to bottom. The top two cards are drawn; one of them is discarded at random, and the other is inserted back at the bottom of the deck. This process is repeated until only one card remains in the deck. Compute the expected value of the label of the remaining card.

Proposed by: Albert Wang

Answer: $\boxed{\frac{467}{8}}$

Solution 1: Note that we can just take averages: every time you draw one of two cards, the EV of the resulting card is the average of the EVs of the two cards. This average must be of the form

$$2^{\bullet} \cdot 1 + 2^{\bullet} \cdot 2 + 2^{\bullet} \cdot 3 + \cdots + 2^{\bullet} \cdot 100$$

where the $2^{\bullet}$s add up to 1. Clearly, the cards further down in the deck get involved in one less layer of averaging, and therefore 1 through 72 are weighted 2^{-7} while the rest are weighted 2^{-6}. To compute the average now, we just add it up to get $\boxed{\frac{467}{8}}$.

Solution 2: We see that in a deck with 2^n cards, that after repeating the process 2^{n-1} times, that each card has a chance of $\frac{1}{2}$ of remaining in the deck. This means that the average of the cards in the deck doesn't change between 2^n by 2^{n-1} cards. Thus, by repeating this process, we determine that the expected value of the last card is the average of all cards whenever we start with 2^n cards.

Suppose we instead start with $2^7 = 128$ cards in the following order:

$$73, 73, 74, 74, \ldots, 100, 100, 1, 2, 3, \ldots, 72.$$

Thus, after 28 steps, we will be left with the original configuration. Since a power of 2 cards are in the deck, we expect that the final card will be the average of these numbers. This is $\boxed{\frac{467}{8}}$.

28. **[14]** Given that the 32-digit integer

$$64\,312\,311\,692\,944\,269\,609\,355\,712\,372\,657$$

is the product of 6 consecutive primes, compute the sum of these 6 primes.

Proposed by: Derek Liu

Answer: $\boxed{1200974}$

Solution: Because the product is approximately $64 \cdot 10^{30}$, we know the primes are all around 200000. Say they are $200000 + x_i$ for $i = 1, \ldots, 6$.

By expanding $\prod_{i=1}^{6}(200000 + x_i)$ as a polynomial in 200000, we see that

$$31231 \cdot 10^{25} = 200000^5(x_1 + \cdots + x_6)$$

plus the carry from the other terms. Note that $31231 = 975 \cdot 32 + 31$, so $x_1 + \cdots + x_6 \leq 975$.

Thus,

$$16(x_1x_2 + x_1x_3 + \cdots + x_5x_6) \leq 16 \cdot \frac{5}{12}(x_1 + \cdots + x_6)^2 < \frac{20}{3} \cdot 1000^2 < 67 \cdot 10^5,$$

so the carry term from $200000^4(x_1x_2 + \cdots + x_5x_6)$ is at most $67 \cdot 10^{25}$. The other terms have negligible carry, so it is pretty clear $x_1 + \cdots + x_6 > 972$, otherwise the carry term would have to be at least

$$31231 \cdot 10^{25} - 200000^5(972) = 127 \cdot 10^{25}.$$

It follows that $x_1 + \cdots + x_6$ lies in $[973, 975]$, so the sum of the primes, $6 \cdot 200000 + (x_1 + \cdots + x_6)$, lies in $[1200973, 1200975]$.

As these primes are all greater than 2, they are all odd, so their sum is even. Thus it must be $\boxed{1200974}$.

29. **[16]** For each prime p, a polynomial $P(x)$ with rational coefficients is called *p-good* if and only if there exist three integers a, b, and c such that $0 \le a < b < c < \frac{p}{3}$ and p divides all the numerators of $P(a)$, $P(b)$, and $P(c)$, when written in simplest form. Compute the number of ordered pairs (r, s) of rational numbers such that the polynomial $x^3 + 10x^2 + rx + s$ is p-good for infinitely many primes p.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{12}$

Solution: By Vieta, the sum of the roots is $-10 \pmod{p}$. However, since the three roots are less than $p/3$, it follows that the roots are $(p-a')/3$, $(p-b')/3$, $(p-c')/3$, where there are finitely many choices $a' < b' < c'$. By pigeonhole, one choice, say (u, v, w) must occur for infinitely many p. We then get that the roots of P are $-u/3$, $-v/3$, and $-w/3$. Moreover, we must have that u, v, w are all $1 \pmod 3$ or all $2 \pmod 3$, and by Vieta, we have $u + v + w = 30$.

The polynomial is then uniquely determined by u, v, w. Thus, it suffices to count triples $u < v < w$ of positive integers such that u, v, w are all $1 \pmod 3$ or all $2 \pmod 3$ and that $u + v + w = 30$. It's not very hard to list them all now.

When $u, v, w \equiv 1 \pmod 3$, there are 7 triples: $(1, 4, 25)$, $(1, 7, 22)$, $(1, 10, 19)$, $(1, 13, 16)$, $(4, 7, 19)$, $(4, 10, 16)$, and $(7, 10, 13)$.

When $u, v, w \equiv 2 \pmod 3$, there are 5 triples: $(2, 5, 23)$, $(2, 8, 20)$, $(2, 11, 17)$, $(5, 8, 17)$, and $(5, 11, 14)$.

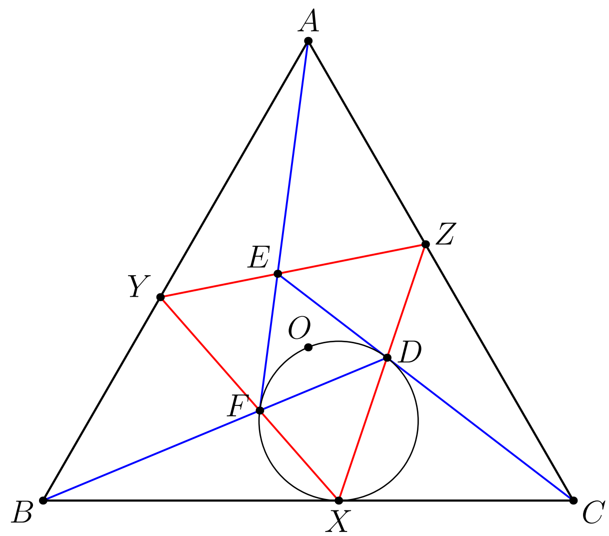
Hence, the answer is $7 + 5 = \boxed{12}$.

30. **[16]** Let ABC be an equilateral triangle with side length 1. Points D, E, F lie inside triangle ABC such that A, E, F are collinear, B, F, D are collinear, C, D, E are collinear, and triangle DEF is equilateral. Suppose that there exists a unique equilateral triangle XYZ with X on side $\overline{BC}$, Y on side $\overline{AB}$, and Z on side $\overline{AC}$ such that D lies on side $\overline{XZ}$, E lies on side $\overline{YZ}$, and F lies on side $\overline{XY}$. Compute AZ.

Proposed by: Jaedon Whyte, Maxim Li

Answer: $\boxed{\frac{1}{1+\sqrt[3]{2}}}$

Solution:



First, note that point X can be constructed from intersection of $\odot(DOF)$ and side $\overline{BC}$. Thus, if there is a unique equilateral triangle, then we must have that $\odot(DOF)$ is tangent to $\overline{BC}$. Furthermore, $\odot(DOF)$ is tangent to DE, so by equal tangents, we have $CD = CX$.

We now compute the answer. Let $x = AZ = CX = CD = BF$. Then, by power of point,

$$BF \cdot BD = BX^2 \implies BD = \frac{(1-x)^2}{x}.$$

Thus, by law of cosine on $\triangle BDC$, we have that

$$\begin{aligned} x^2 + \left(\frac{(1-x)^2}{x}\right)^2 + x \cdot \frac{(1-x)^2}{x} &= 1 \\ x^2 + \frac{(1-x)^4}{x^2} + (1-x)^2 &= 1 \\ \frac{(1-x)^4}{x^2} &= 2x(1-x) \\ \frac{1-x}{x} &= \sqrt[3]{2} \\ x &= \boxed{\frac{1}{1+\sqrt[3]{2}}}. \end{aligned}$$

31. **[16]** Ash and Gary independently come up with their own lineups of 15 fire, grass, and water monsters. Then, the first monster of both lineups will fight, with fire beating grass, grass beating water, and water beating fire. The defeated monster is then substituted with the next one from their team's lineup; if there is a draw, both monsters get defeated.

Gary completes his lineup randomly, with each monster being equally likely to be any of the three types. Without seeing Gary's lineup, Ash chooses a lineup that maximizes the probability p that his monsters are the last ones standing. Compute p.

Proposed by: Albert Wang

Answer: $\boxed{1 - \frac{2^{15}}{3^{15}}}$

Solution: First, we show Ash cannot do better. Notice there is a $\frac{2^{15}}{3^{15}}$ chance that Gary's i-th monster ties or defeats Ash's i-th monster for each i. If this is the case, Ash cannot win, as Ash's i-th monster will always be defeated by Gary's i-th monster, if not sooner. Thus, Ash wins with probability at most $1-\frac{2^{15}}{3^{15}}$. It remains to show this is achievable.

Ash uses the lineup fire-grass-water repeated 5 times. Then, none of Gary's monsters can defeat more than one monster in Ash's lineup, so Ash will win unless Gary manages to take down exactly one monster with each of his. In particular, this means the i-th monster Gary has must tie or defeat Ash's i-th monster, which occurs with $\frac{2}{3}$ chance with each i. Thus this construction achieves the answer of $\boxed{1-\frac{2^{15}}{3^{15}}}$.

32. **[16]** Over all pairs of complex numbers (x, y) satisfying the equations

$$x+2y^2=x^4 \quad \text{and} \quad y+2x^2=y^4,$$

compute the minimum possible real part of x.

Proposed by: Jaedon Whyte

Answer: $\boxed{\sqrt[3]{\frac{1-\sqrt{33}}{2}}}$

Solution 1: Note the following observations:

(a) if (x, y) is a solution then $(\omega x, \omega^2 y)$ is also a solution if $\omega^3=1$ and $\omega \neq 1$.

(b) we have some solutions (x, x) where x is a solution of $x^4-2x^2-x=0$.

These are really the only necessary observations and the first does not need to be noticed immediately. Indeed, we can try to solve this directly as follows: first, from the first equation, we get $y^2=\frac{1}{2}(x^4-x)$, so inserting this into the second equation gives

$$\begin{aligned}
\frac{1}{4}(x^4-x)^2-2x^2&=y\\
\left((x^4-x)^2-8x^2\right)^2-8x^4+8x&=0\\
(x^8-2x^5-7x^2)^2-8x^4+8x&=0\\
\underbrace{x^{16}+\cdots+41x^4+8x}_{P(x)}&=0
\end{aligned}$$

By the second observation, we have that $x(x^3-2x-1)$ should be a factor of P. The first observation gives that $(x^3-2\omega x-1)(x^3-2\omega^2 x-1)$ should therefore also be factor. Now $(x^3-2\omega x-1)(x^3-2\omega^2 x-1)=x^6+2x^4-2x^3+4x^2-2x+1$ since ω and ω^2 are roots of x^2+x+1. So now we see that the last two terms of the product of all of these is $-5x^4-x$. Hence the last two terms of the polynomial we get after dividing out should be $-x^3-8$, and given what we know about the degree and the fact that everything is monic, the quotient must be exactly x^6-x^3-8 which has roots being the cube roots of the roots to x^2-x-8, which are $\sqrt[3]{\frac{1\pm\sqrt{33}}{2}}$. Now x^3-2x-1 is further factorable as $(x-1)(x^2-x-1)$ with roots $1, \frac{1\pm\sqrt{5}}{2}$ so it is not difficult to compare the real parts of all roots of P, especially since 5 are real and non-zero, and we have that $\operatorname{Re}(\omega x)=-\frac{1}{2}x$ if $x \in \mathbb{R}$. We conclude that the smallest is $\boxed{\sqrt[3]{\frac{1-\sqrt{33}}{2}}}$.

Solution 2: Subtracting the second equation from the first, we get:

$$\begin{aligned}
(y+2x^2)-(x+2y^2)&=y^4-x^4 \implies\\
(x-y)+2(x^2-y^2)&=(x^2-y^2)(x^2+y^2) \implies
\end{aligned}$$

$$(x-y)(1-(x+y)(x^2+y^2+2))=0$$

Subtracting y times the first equation from x times the second, we get:

$$(xy+2y^3)-(xy+2x^3)=x^4y-xy^4 \implies$$
$$2(y^3-x^3)=xy(x^3-y^3) \implies$$
$$(x^3-y^3)(2+xy)=0$$

Subtracting y^2 times the second equation from x^2 times the first, we get:

$$(x^3+2x^2y^2)-(y^3+2x^2y^2)=x^6-y^6 \implies$$
$$x^3-y^3=(x^3+y^3)(x^3-y^3) \implies$$
$$(x^3-y^3)(1-x^3-y^3)=0$$

We have three cases.

Case 0. $x=0$ Thus, $(x,y)=(0,0)$ is the only valid solution.

Case 1. $x=\omega y$ for some third root of unity ω. Thus, $y^2=\omega^4x^2=\omega x^2$

$$x+2y^2=x^4 \implies$$
$$x+2\omega x^2=x^4 \implies$$
$$x(1+\omega)(2-\omega x^2)=1$$

Note that $x=-\omega$ is always a solution to the above, and so we can factor as:

$$x^3+2(1+\omega)x-1=0$$
$$(x+\omega)(x^2-\omega x-\omega^2)=0$$

and so the other solutions are of the form:

$$x=\frac{1\pm\sqrt{5}}{2}\cdot\omega$$

for the third root of unity. The minimum real part in this case is $-\frac{1+\sqrt{5}}{2}$ when $\omega=1$.

Case 2. Since $x^3-y^3\neq 0$, we have $xy=-2$ and $x^3+y^3=1$.

Thus, $x^3-y^3=\sqrt{(x^3+y^3)^2-4(xy)^2}=\pm\sqrt{33} \implies x^3=\left(\frac{1\pm\sqrt{33}}{2}\right)$

This yields the minimum solution of $x=\boxed{\left(\frac{1-\sqrt{33}}{2}\right)^{1/3}}$ as desired. This is satisfied by letting $y=\left(\frac{1+\sqrt{33}}{2}\right)^{1/3}$.

33. **[20]** Let p denote the proportion of teams, out of all participating teams, who submitted a negative response to problem 5 of the Team round (e.g. "there are no such integers"). Estimate $P=\lfloor 10000p\rfloor$. An estimate of E earns $\max(0,\lfloor 20-|P-E|/20\rfloor)$ points.

If you have forgotten, problem 5 of the Team round was the following: "Determine, with proof, whether there exist positive integers x and y such that $x+y$, x^2+y^2, and x^3+y^3 are all perfect squares."

Proposed by: Arul Kolla

Answer: $\boxed{5568}$

Solution: Of the 88 teams competing in this year's Team round, 49 of them answered negatively, 9 (correctly) provided a construction, 16 answered ambiguously or did not provide a construction, and the remaining 14 teams did not submit to problem 5. Thus $p=\frac{49}{88}\approx 0.5568$.

34. **[20]** Estimate the number of positive integers $n \leq 10^6$ such that n^2+1 has a prime factor greater than n.

Submit a positive integer E. If the correct answer is A, you will receive $\max\left(0, \left\lfloor 20 \cdot \min\left(\frac{E}{A}, \frac{10^6-E}{10^6-A}\right)^5 + 0.5 \right\rfloor\right)$ points.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{757575}$

Solution: Let N denote 10^6. We count by summing over potential prime factors p.

For any prime $p > 2$, we have that $p \mid n^2+1$ for two values of n if $p \equiv 1 \pmod 4$, and zero values otherwise. Pretending these values are equally likely to be any of $1, \ldots, p$, we expect the number of n corresponding to a $1 \pmod 4$ prime to be $\min\left(2, \frac{2N}{p}\right)$.

The number of primes up to x is, by the Prime Number Theorem $\frac{x}{\log x}$. Assuming around half of the prime numbers are $1 \pmod 4$, we on average expect some x to be a $1 \pmod 4$ prime $\frac{1}{2\log x}$ of the time. Approximating by an integral over potential primes x from 1 to N^2, using our approximations, gives

$$\int_1^{N^2} \min\left(2, \frac{2N}{x}\right) \cdot \frac{dx}{2\log x}.$$

We now approximately calculate this integral as follows:

$$\begin{aligned}
\int_1^{N^2} \min\left(2, \frac{2N}{x}\right) \cdot \frac{dx}{2\log x} &= \int_1^N \frac{dx}{\log x} + \int_N^{N^2} \frac{N}{x\log x} dx \\
&\approx \frac{N}{\log N} + N(\log\log(N^2) - \log\log N) \\
&= \frac{N}{\log N} + N\log 2.
\end{aligned}$$

Here, for the first integral, we estimate $\log x$ on $[1, N]$ by $\log N$, and for the second integral, we use that the antiderivative of $\frac{1}{x\log x}$ is $\log\log x$.

Using $\log 2 \approx 0.7$, one can estimate

$$\log N = 2\log 1000 \approx 20\log 2 \approx 14$$

giving a final estimate of

$$10^6/14 + 10^6 \cdot 0.7 = 771428.$$

This estimate yields a score of 15. If one uses the closer estimate $\log 2 \approx 0.69$, one gets the final estimate of 761428, yielding a score of 18.

Here is a code using `sympy` to calculate the final answer:

```python
from sympy.ntheory import factorint
cnt = 0
for n in range(1,10**6+1):
    if max(factorint(n**2+1, multiple=True)) > n:
        cnt += 1
print(cnt)
```

35. **[20]** Barry picks infinitely many points inside a unit circle, each independently and uniformly at random, $P_1, P_2, \ldots$. Compute the expected value of N, where N is the smallest integer such that P_{N+1} is inside the convex hull formed by the points $P_1, P_2, \ldots, P_N$.

Submit a positive real number E. If the correct answer is A, you will receive $\lfloor 100 \cdot \max(0.2099 - |E - A|, 0) \rfloor$ points.

Proposed by: Albert Wang, Rishabh Das

Answer: $\boxed{6.54}$

Solution: Clearly, $N \geq 3$, and let's scale the circle to have area 1. We can see that the probability to not reach $N = 4$ is equal to the probability that the fourth point is inside the convex hull of the past three points. That is, the probability is just one minus the expected area of those N points. The area of this turns out to be really small, and is around 0.074, and so $(1 - 0.074)$ of all sequences of points make it to $N = 4$. The probability to reach to the fifth point from there should be around $(1 - 0.074)(1 - 0.074 \cdot 2)$, as any four points in convex configuration can be covered with 2 triangles. Similarly, the chance of reaching $N = 6$ should be around $(1 - 0.074)(1 - 0.074 \cdot 2)(1 - 0.074 \cdot 3)$, and so on. Noting that our terms eventually decay to zero around term $1/0.074 = 13$, our answer should be an underestimate. In particular, we get

$$3 + (1 - 0.074)(1 + (1 - 0.074 \cdot 2)(1 + (1 - 0.074 \cdot 3)(1 + \cdots))) \approx 6.3.$$

Guessing anything slightly above this lower bound should give a positive score.

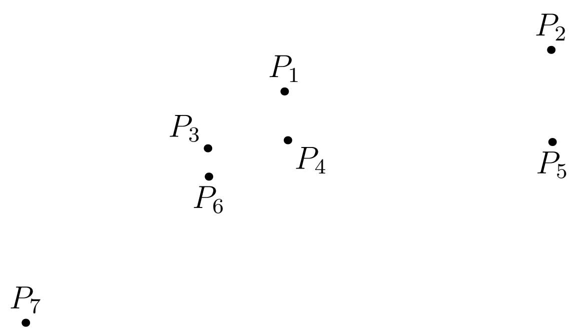
Here is a Python code that simulates the result.

```
from random import randrange,getrandbits
import itertools, math
from tqdm import tqdm
import numpy as np

DEBUG = False

def unit_circle_pt():
    while True:
        x = randrange(-(2**32),2**32+1)
        y = randrange(-(2**32),2**32+1)
        if x*x + y*y < 2**64:
            return (x,y)

def area_of_triangle(p1, p2, p3):
    return abs((p2[0] - p1[0])*(p3[1] - p2[1]) - (p2[1] - p1[1])*(p3[0] - p2[0]))

def pt_inside_polygon(point, polygon):
    # point is a pair
    # polygon is an angle-sorted list of points that are the vertices of a convex polygon in
                                                        some order

    # area of polygon

    # plot the polygon and the point
    if DEBUG:
        import matplotlib.pyplot as plt
        # plot the big circle
        circle = plt.Circle((0,0), 2**32, color='b', fill=False)
        # fix view to circle
        plt.xlim(-2**32,2**32)
        plt.ylim(-2**32,2**32)
        plt.gca().add_artist(circle)
        # make the window to scale
        plt.gca().set_aspect('equal', adjustable='box')
        plt.plot([x for x,_ in polygon], [y for _,y in polygon])
        plt.scatter([point[0]], [point[1]])
        plt.show()

    area = 0
    for i in range(1, len(polygon)-1):
        # add on area between points 0, i, i+1
        area += area_of_triangle(polygon[0], polygon[i], polygon[i+1])
```

```
    # point is inside polygon if the area of the triangles formed by the point and each edge
                                                 of the polygon sum to the area of the
                                                polygon
    area_sum = 0
    for i in range(len(polygon)):
        # add on area between points point, polygon[i], polygon[(i+1)%len(polygon)]
        area_sum += area_of_triangle(point, polygon[i], polygon[(i+1)%len(polygon)])

    return area_sum == area

def convex_hull(points):
    # sort by x, then y
    points = sorted(points, key=lambda x: (x[0], x[1]))
    # graham scan

    # find the lowest point
    lowest = points[0]
    for p in points:
        if p[1] < lowest[1]:
            lowest = p
    # sort by angle
    points = sorted(points, key=lambda x: (math.atan2(x[1]-lowest[1], x[0]-lowest[0]), -x[1]
                                                , x[0]))
    # remove duplicates
    points = list(k for k,_ in itertools.groupby(points))
    # stack to hold the points
    stack = []
    for p in points:
        while len(stack) > 1 and (stack[-1][0]-stack[-2][0])*(p[1]-stack[-2][1]) - (stack[-1
                                                    ][1]-stack[-2][1])*(p[0]-stack[-2][
                                                    0]) <= 0:
            stack.pop()
        stack.append(p)
    return stack

def pulse(horizon=1000):
    cur = [unit_circle_pt() for _ in range(3)]

    for N in range(3, horizon):
        pt = unit_circle_pt()
        if pt_inside_polygon(pt, cur):
            return N
        cur = convex_hull(cur + [pt])

trials = 1000
blocks = 100000
cur_trials = 0
cur_sum = 0
results = []

for block in range(trials):
    for _ in tqdm(range(blocks)):
        results.append(pulse())
    cur_trials += blocks
    mean = np.mean(results)
    stddev = np.std(results)
    z = 5.0
    ci = (mean - z*stddev/np.sqrt(cur_trials), mean + z*stddev/np.sqrt(cur_trials))
    print(block+1, mean, stddev, ci)
```

36. **[20]** Let ABC be a triangle. The following diagram contains points $P_1, P_2, \ldots, P_7$, which are the following triangle centers of triangle ABC in some order:

- the incenter I;
- the circumcenter O;
- the orthocenter H;
- the symmedian point L, which is the intersections of the reflections of B-median and C-median across angle bisectors of $\angle ABC$ and $\angle ACB$, respectively;
- the Gergonne point G, which is the intersection of lines from B and C to the tangency points of the incircle with $\overline{AC}$ and $\overline{AB}$, respectively;
- the Nagel point N, which is the intersection of line from B to the tangency point between B-excircle and $\overline{AC}$, and line from C to the tangency point between C-excircle and $\overline{AB}$; and
- the Kosnita point K, which is the intersection of lines from B and C to the circumcenters of triangles AOC and AOB, respectively.



Note that the triangle ABC is not shown. Compute which triangle centers $\{I, O, H, L, G, N, K\}$ corresponds to P_k for $k \in \{1, 2, 3, 4, 5, 6, 7\}$.

Your answer should be a seven-character string containing I, O, H, L, G, N, K, or X for blank. For instance, if you think $P_2 = H$ and $P_6 = L$, you would answer $XHXXXLX$. If you attempt to identify $n > 0$ points and get them **all** correct, then you will receive $\lceil (n-1)^{5/3} \rceil$ points. Otherwise, you will receive 0 points.

Proposed by: Kevin Zhao, Pitchayut Saengrungkongka

Answer: $\boxed{KOLINGH}$

Solution: Let G' be the centroid of triangle ABC. Recall the following.

- Points O, G', H lie on **Euler's line** of $\triangle ABC$ with $OG' : G'H = 1 : 2$.
- Points I, G', N lie on **Nagel's line** of $\triangle ABC$ with $IG' : G'N = 1 : 2$.

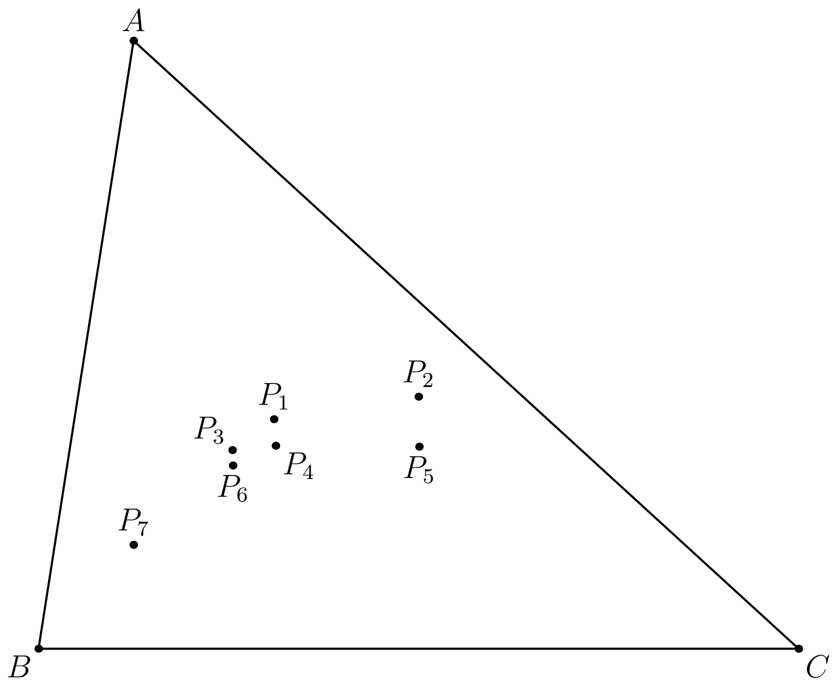
Thus, $OI \parallel HN$ with $OI : HN = 1 : 2$. Therefore, we can detect parallel lines with ratio $2 : 1$ in the figure. The only possible pairs are $P_2P_4 \parallel P_7P_5$. Therefore, there are two possibilities: (P_2, P_7) and (P_4, P_5) must be (O, H) and (I, N) in some order. Intuitively, H should be further out, so it's not unreasonable to guess that $P_2 = O$, $P_7 = H$, $P_4 = I$, and $P_5 = N$. Alternatively, perform the algorithm below with the other case to see if it fails.

To identify the remaining points, we recall that the isogonal conjugate of G and N both lie on OI (they are insimilicenter and exsimilicenter of incircle and circumcircle, respectively). Thus, H, G, N, I lie on isogonal conjugate of OI, known as the **Feuerbach's Hyperbola**. It's also known that OI is tangent to this line, and this hyperbola have perpendicular asymptotes.

Using all information in the above paragraph, we can eyeball a rectangular hyperbola passing through H, G, N, I and is tangent to OI. It's then not hard to see that $P_6 = G$.

Finally, we need to distinguish between symmedian and Kosnita points. To do that, recall that Kosnita point is isogonal conjugate of the nine-point center (not hard to show). Thus, H, L, K, O lies on isogonal conjugate of OH, which is the **Jerabek's Hyperbola**. One can see that H, L, K, O lies on the same branch. Moreover, they lie on this hyperbola in this order because the isogonal conjugates (in order) are O, centroid, nine-point center, and H, which lies on OH in this order. Using this fact, we can identity $P_5 = L$ and $P_1 = K$, completing the identification.

The following is the diagram with the triangle ABC.



Here is the Asymptote code that generates the diagram in the problem.

```
import olympiad;
import geometry;
size(7.5cm);
pair A = (0.5,3.2);
pair B = (0,0);
pair C = (4,0);
pair O = circumcenter(triangle(A,B,C));
pair H = orthocentercenter(triangle(A,B,C));
pair L = symmedian(triangle(A,B,C));
pair Ge = gergonne(triangle(A,B,C));
pair I = incenter(triangle(A,B,C));
pair Na = A+B+C - 2I;
pair K = extension(A, circumcenter(B,O,C), B, circumcenter(A,O,C));

dot("$P_6$",Ge,dir(-90));
dot("$P_2$",O,dir(90));
dot("$P_7$",H,dir(90));
dot("$P_3$",L,dir(135));
dot("$P_4$",I,dir(-45));
dot("$P_5$",Na,dir(-90));
dot("$P_1$",K,dir(90));
```