HMMT February 2024

February 17, 2024

Team Round

1. **[20]** Let $a_1, a_2, a_3, \ldots, a_{100}$ be integers such that
$$\frac{a_1^2 + a_2^2 + a_3^2 + \cdots + a_{100}^2}{a_1 + a_2 + a_3 + \cdots + a_{100}} = 100.$$
Determine, with proof, the maximum possible value of a_1.

2. **[25]** Nine distinct positive integers summing to 74 are put into a 3×3 grid. Simultaneously, the number in each cell is replaced with the sum of the numbers in its adjacent cells. (Two cells are adjacent if they share an edge.) After this, exactly four of the numbers in the grid are 23. Determine, with proof, all possible numbers that could have been originally in the center of the grid.

3. **[25]** Let ABC be a scalene triangle and M be the midpoint of BC. Let X be the point such that $CX \parallel AB$ and $\angle AMX = 90^\circ$. Prove that AM bisects $\angle BAX$.

4. **[30]** Each lattice point with nonnegative coordinates is labeled with a nonnegative integer in such a way that the point $(0, 0)$ is labeled by 0, and for every $x, y \geq 0$, the set of numbers labeled on the points (x, y), $(x, y+1)$, and $(x+1, y)$ is $\{n, n+1, n+2\}$ for some nonnegative integer n. Determine, with proof, all possible labels for the point $(2000, 2024)$.

5. **[40]** Determine, with proof, whether there exist positive integers x and y such that $x+y$, x^2+y^2, and x^3+y^3 are all perfect squares.

6. **[45]** Let $\mathbb{Q}$ be the set of rational numbers. Given a rational number $a \neq 0$, find, with proof, all functions $f : \mathbb{Q} \to \mathbb{Q}$ satisfying the equation
$$f(f(x) + ay) = af(y) + x$$
for all $x, y \in \mathbb{Q}$.

7. **[50]** Let $ABCDEF$ be a regular hexagon with P as a point in its interior. Prove that of the three values $\tan \angle APD$, $\tan \angle BPE$, and $\tan \angle CPF$, two of them sum to the third one.

8. **[50]** Let P be a point in the interior of quadrilateral $ABCD$ such that the circumcircles of triangles PDA, PAB, and PBC are pairwise distinct but congruent. Let the lines AD and BC meet at X. If O is the circumcenter of triangle XCD, prove that $OP \perp AB$.

9. **[55]** On each cell of a 200×200 grid, we place a car, which faces in one of the four cardinal directions. In a move, one chooses a car that does not have a car immediately in front of it, and slides it one cell forward. If a move would cause a car to exit the grid, the car is removed instead. The cars are placed so that there exists a sequence of moves that eventually removes all the cars from the grid. Across all such starting configurations, determine the maximum possible number of moves to do so.

10. **[60]** Across all polynomials P such that $P(n)$ is an integer for all integers n, determine, with proof, all possible values of $P(i)$, where $i^2 = -1$.