HMIC 2024

April 14-21, 2024

1. **[6]** In an empty 100×100 grid, 300 cells are colored blue, 3 in each row and each column. Compute the largest positive integer k such that you can always recolor k of these blue cells red so that no contiguous 2×2 square has four red cells.

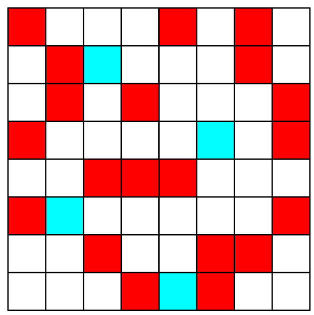
   Proposed by: Arul Kolla

   Answer: $\boxed{250}$

   Solution: We first prove the lower bound. We can recolor all three blue cells in odd rows, and the first and third blue cells in even rows.

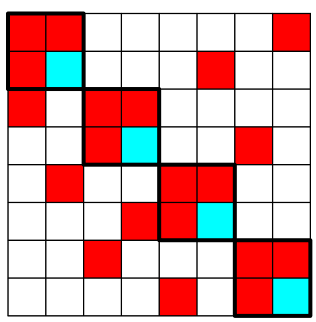
   A 2×2 square must cover two adjacent cells in an even row. Therefore, it's impossible for there to exist a red 2×2 square in our recoloring, because coloring the first and third cell in each even row guarantees that no two adjacent cells are both red in that row.

   This colors $50 \times 3 + 50 \times 2 = 250$ cells.

   For the upper bound, consider the configuration with 50 disjoint 2×2 squares colored blue along the diagonal. Then, color the remaining 100 cells more or less arbitrarily (for instance, the other main diagonal).

   For each 2×2 blue square, we can color at most 3 of those cells red. Therefore, we can recolor at most $3 \cdot 50 + 100 = 250$ cells.

2. **[7]** Suppose that a, b, c, and d are real numbers such that $a + b + c + d = 8$. Compute the minimum possible value of
$$20(a^2 + b^2 + c^2 + d^2) - \sum_{\text{sym}} a^3 b,$$
   where the sum is over all 12 symmetric terms.

   Proposed by: Derek Liu

Answer: $\boxed{112}$

Solution: Observe that

$$\sum_{\text{sym}} a^3 b = \sum_{\text{cyc}} a \cdot \sum_{\text{cyc}} a^3 - \sum_{\text{cyc}} a^4 = 8\sum_{\text{cyc}} a^3 - \sum_{\text{cyc}} a^4,$$

so

$$\begin{aligned}
20\sum_{\text{cyc}} a^2 - \sum_{\text{sym}} a^3 b &= \sum_{\text{cyc}} a^4 - 8\sum_{\text{cyc}} a^3 + 20\sum_{\text{cyc}} a^2 \\
&= \sum_{\text{cyc}} (a^4 - 8a^3 + 20a^2) \\
&= 16(8) - 4(4) + \sum_{\text{cyc}} (a^4 - 8a^3 + 20a^2 - 16a + 4) \\
&= 112 + \sum_{\text{cyc}} (a^2 - 4a + 2)^2 \\
&\geq 112.
\end{aligned}$$

Equality is achieved when $(a, b, c, d) = (2 + \sqrt{2}, 2 + \sqrt{2}, 2 - \sqrt{2}, 2 - \sqrt{2})$ and permutations. This can be checked by noting that for all $x \in \{a, b, c, d\}$, we have $x^2 - 4x + 2 = 0$, so equality holds in the final step, and for this assignment of variables we have $a + b + c + d = 8$.

3. **[9]** Let S be a set of nonnegative integers such that

- there exist two elements a and b in S such that $a, b > 1$ and $\gcd(a, b) = 1$; and
- for any (not necessarily distinct) element x and nonzero element y in S, both xy and the remainder when x is divided by y are in S.

Prove that S contains every nonnegative integer.

Proposed by: Jacob Paltrowitz

Solution: Assume $a < b$. Note that we can get $1 \in S$ via the Euclidean algorithm, and $0 \in S$ from $a \bmod 1$. Suppose $(a, b) \neq (2, 3)$. We will show that there exists $c, d \in S$ with $1 < c \leq a$ and $1 < d \leq b$, with $\gcd(c, d) = 1$ and at least one of $c \neq a$ and $d \neq b$ is true. This will show that $2, 3 \in S$ by repeatedly applying this sequence of operations. Assume $a > 2$.

Unless $b \bmod a = 1$, $(b \bmod a, a)$ works. Thus, assume $b \equiv 1 \pmod{a}$. If we can construct any integer x such that $x \not\equiv 0, 1 \mod a$, then we can take $(x \bmod a, a)$; this will be our new goal. Suppose that such an x is not constructable.

Suppose $a^k = qb^\ell + r$, with $r < a^k$. We must have $r \equiv 0, 1 \pmod{a}$. Taking both sides mod a, this means that $0 \equiv q + r \pmod{a}$, so $q \equiv 0, -1 \pmod{a}$. Thus, $\left\lfloor \frac{a^k}{b^\ell} \right\rfloor \equiv 0, -1 \pmod{a}$, for all $k, \ell \in \mathbb{N}$. Take $\ell = \lfloor k \log_b a \rfloor$, so $\left\lfloor \frac{a^k}{b^\ell} \right\rfloor$ is the first digit in the base b representation of a^k. We can take k such that $0 \leq \{k \log_b a\} < \log_b 2$, where $\{\cdot\}$ represents the fractional part operation, since $\log_b a$ is irrational. Taking such a k, we get the first digit of a^k in base b is 1, which contradicts $\left\lfloor \frac{a^k}{b^\ell} \right\rfloor \equiv 0, -1 \pmod{a}$.

Thus, if $a > 2$ then we have shown what we wanted. Suppose $a = 2$. Let $2^k < b < 2^{k+1}$ for some k. If $b \neq 3$, then either $b \mod 2^k \neq 1$ or $2^{k+1} \mod b \neq 1$, and both numbers are less than b, so either $(2, b \bmod 2^k)$ or $(2, 2^{k+1} \mod b)$ works.

We have shown $2, 3 \in S$. Then $4 \in S$, and $5 = 32 \mod 27 \in S$. Suppose $n \geq 6$ is the smallest integer that isn't in S. Then $n - 4, n - 2, n - 1 \in S$, so $n^2 - 5n + 4, n^2 - 4n + 4$ are both in S. A computation shows that $n^2 - 4n + 4 < 2(n^2 - 5n + 4)$, and so

$$n^2 - 4n + 4 \mod (n^2 - 5n + 4) = n \in S,$$

a contradiction. Thus, every single nonnegative integer is in S.

4. **[9]** Given a positive integer n, let $[n] = \{1, 2, \ldots, n\}$.

   - Let a_n denote the number of functions $f : [n] \to [n]$ such that $f(f(i)) \geq i$ for all i.
   - Let b_n denote the number of ordered set partitions of $[n]$, i.e., the number of ways to pick an integer k and an ordered k-tuple of pairwise disjoint nonempty sets $(A_1, \ldots, A_k)$ whose union is $[n]$.

   Prove that $a_n = b_n$.

   Proposed by: Derek Liu

   Solution: It suffices to define a bijection between the two types of objects in the problem for each n. We'll be a bit more general and define a recursive bijection from ordered set partitions of $S \subseteq [n]$ to functions $f : S \to S$ described in the problem as follows:

   > If S is empty, return the trivial function $f : \varnothing \to \varnothing$.
   >
   > Otherwise, consider some ordered set partition $A_1, \ldots, A_k$ of $S \subseteq [n]$, and take A_1. Order the elements of S as $s_1 < s_2 < \cdots < s_\ell$. Now, let s_m denote the largest element of A_1. Set $f(s_1) = s_m$, and for each $s_i \in A_1 \setminus \{s_m\}$, set $f(s_{i+1}) = s_1$.
   >
   > Then, let $A_1' = \{s_{i+1} : s_i \in A_1, s_i \neq s_m\} \cup \{s_1\}$ to be the elements of $A_1 \setminus \{s_m\}$ shifted over by 1 index with s_1 added in. Create a new ordered set partition of $S \setminus A_1'$ by taking the relative ordering of entries of $A_2, \ldots A_k$, and relabelling them to match the elements of $S \setminus A_1'$. To verify that this reordering can be done, we need to check that the sizes $|A_1'| = |A_1|$. However, $|\{s_{i+1} : s_i \in A_1, s_i \neq s_m\}| = |A_1| - 1$, and s_1 is not in this set, so the new partition is valid.
   >
   > Let $f' : S \setminus A_1' \to S \setminus A_1'$ be the output of this process recursively applied on the set $S \setminus A_1'$ with the new set partition. We can then naturally merge our values of f, f' such that if $x \in A_1'$, $f(x)$ is defined as above, and if $x \in S \setminus A_1'$, we take $f(x) := f'(x)$.

   We now need to show that this process is well-defined, i.e. any input partition causes it to output a function $f : S \to S$ satisfying the desired property, and that it has a well-defined inverse.

   First, we observe that the process always outputs a valid function. We can do this by induction; it is vacuously true on the base case $S = \varnothing$. Then, it suffices to show that if the recursive step output $f' : S \setminus A_1' \to S \setminus A_1'$ satisfies $f'(f'(i)) \geq i$, then adding on the new values of f for A_1' preserves this property. Indeed, we only need to verify it at inputs in A_1', which are s_{i+1} for $s_i \in A_1$ with $i < m$, and also s_1.

   To check this, let $s_i \in A_1$ with $s_i < s_m$. Then

   $$f(f(s_{i+1})) = f(s_1) = s_m \geq s_{i+1}.$$

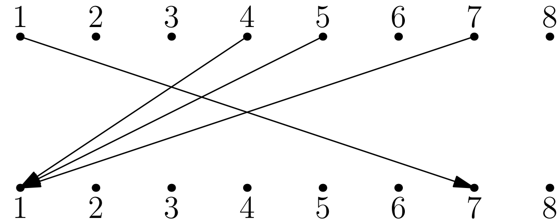
   Similarly, it is always true that $f(f(s_1)) \geq s_1$ since it is the smallest element of S. So, the induction is complete.

   Finally, we need to show that this process is invertible. We do this with another recursive process; start any function $f : S \to S$ satisfying the desired property. We can then define A_1 to be $\{s_{i-1} : s_i \in f^{-1}(s_1), s_i > s_1\} \cup \{f(s_1)\}$ for $f^{-1}(s_1)$ the preimage of s_1 in f. Note that for any $s_i \in S$ with $s_i > s_1$, we do not have $f(s_i) \in f^{-1}(s_1)$ as otherwise $f(f(s_i)) = s_1 < s_i$. Furthermore, no elements of $S \setminus f^{-1}(s_1)$ can map to s_1 due to the definition of a preimage. So, we can restrict f to a function $f' : S \setminus (f^{-1}(s_1) \cup \{s_1\}) \to S \setminus (f^{-1}(s_1) \cup \{s_1\})$, on which we can recurse to obtain a partition $A_2, \ldots, A_k$ and relabel to $S \setminus A_1$ to obtain a partition $A_1, \ldots, A_k$ of S.

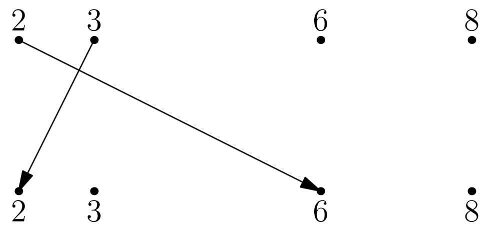
   This process is well-defined, and is clearly an inverse, as desired.

   Below is an example of the bijection for $n = 8$ with the ordered set partition $\{3, 4, 6, 7\} \cup \{1, 5\} \cup \{2, 8\}$:

Step 1
$A_1 = \{3,4,6,7\}$
A_2, A_3: $\{1,5\},\{2,8\}$
Unused elements: $\{2,3,6,8\}$
Relabeled A_2, A_3 : $\{2,6\},\{3,8\}$

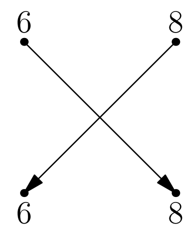


Step 2
$A_2 = \{2,6\}$
A_3: $\{3,8\}$
Unused elements: $\{6,8\}$
Relabeled A_3: $\{6,8\}$

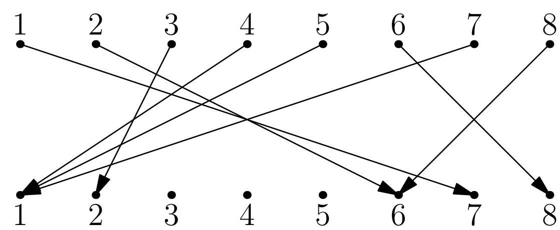


Step 3

$A_3 = \{6,8\}$



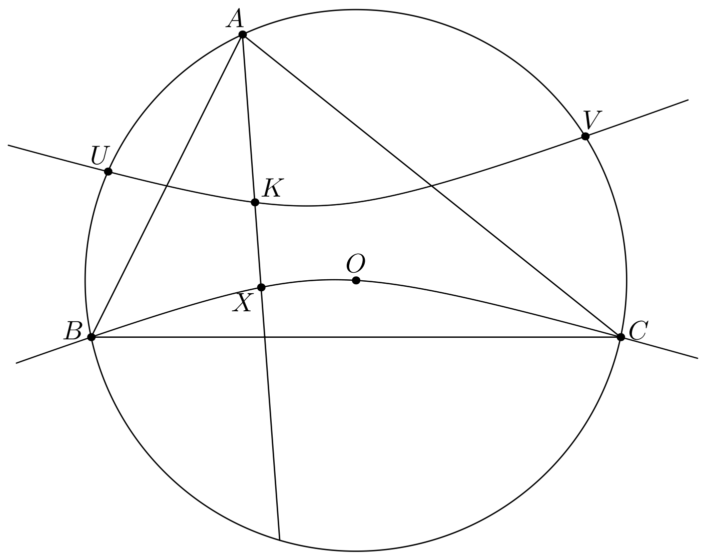
Output function:



5. **[11]** Let ABC be an acute, scalene triangle with circumcenter O and symmedian point K. Let X be the point on the circumcircle of triangle BOC such that $\angle AXO = 90°$. Assume that $X \neq K$. The hyperbola passing through B, C, O, K, and X intersects the circumcircle of triangle ABC at points U and V, distinct from B and C. Prove that UV is the perpendicular bisector of AX.

The symmedian point of triangle ABC is the intersection of the reflections of B-median and C-median across the angle bisectors of $\angle ABC$ and $\angle ACB$, respectively.

Proposed by: Pitchayut Saengrungkongka



Solution 1: Let $\mathcal{H}$ denote the hyperbola, and also recall a well-known fact that A, K, X are collinear. This solution is split into two independent parts.

Proof of $UV \perp AX$.

Let $T = OX \cap BC$, and let OX intersect $\odot(ABC)$ at points Y, Z. Then, we apply Desargues Involution Theorem (DIT) on $BCUV$ and line OX. We get an involution that

- swaps T and $UV \cap OX$;
- swaps $BU \cap OX$ and $CV \cap OX$;
- swaps $BV \cap OX$ and $CU \cap OX$;
- from conic $\odot(ABCUV)$, swaps Y and Z;
- from conic $\mathcal{H}$, swaps O and X.

Thus, we get an involution swapping (O, X) and (Y, Z), and $(T, UV \cap OX)$. Now, notice that since B, O, X, C are concyclic, we have $TY \cdot TZ = TB \cdot TC = TO \cdot TX$, so this involution is an inversion at T. Thus, it maps T to ∞_{OX}. Hence, U, V, ∞_{OX} are collinear, implying that $UV \perp AX$.

Proof of UV bisects AX.

Let M be the midpoint of AX. Let AX intersect BC at D and $\odot(ABC)$ at P. We apply Desargues Involution Theorem (DIT) on $BCUV$ and line AX. We get an involution that

- swaps D and $UV \cap OX$;
- swaps $BU \cap AX$ and $CV \cap AX$;
- swaps $BV \cap AX$ and $CU \cap AX$;
- from conic $\odot(ABCUV)$, swaps A and P;
- from conic $\mathcal{H}$, swaps K and X.

Thus, we have an involution swapping (A, P) and (K, X), and we want to show that it swaps (D, M), or equivalently

$$(A, X; M, P) = (P, K; D, A).$$

The left hand side is $\frac{AM}{XM} / \frac{AP}{XP} = -2$. To show that $(P, K; D, A) = -2$, we take any projective transformation that fixes $\odot(ABC)$ and sends $\triangle ABC$ to an equilateral triangle. This transformation preserves $BB \cap CC$, $CC \cap AA$, and $AA \cap BB$, so it preserves symmedians. The result then become evident.

Remark. Another interesting fact about this hyperbola is that it passes through midpoints of AB and AC. We left the proof of this as an exercise for the reader.

Solution 2: Let AK meet BC at Z, and redefine U, V to be the intersections of perpendicular bisector of AX and $\odot(ABC)$. We will use the following properties of X:

- X lies on AK
- $\triangle BXA \sim \triangle AXC$ with ratio $c : b$
- $\angle BXA = \angle ACX = 180^\circ - A$.

Consider the quadratic function

$$f(P) = \frac{\operatorname{Pow}_{\odot(ABC)}(P)}{\operatorname{Pow}_{\odot(ABC)}(O)} - \frac{d(P, BC)d(P, UV)}{d(O, BC)d(O, UV)}.$$

It is clear (say, in coordinates) that f is the equation of a conic passing through B, C, U, V, and O. Hence, it suffices to show $f(X) = 0$ and $f(K) = 0$.

Since $OX||UV$, we have

$$f(X) = \frac{\operatorname{Pow}_{\odot(ABC)}(X)}{\operatorname{Pow}_{\odot(ABC)}(O)} - \frac{d(X, BC)}{d(O, BC)}.$$

The RHS above is a circle (in coordinates) and is 0 when $X \in \{O, B, C\}$, and is hence 0 for any X on (BOC). Thus, $f(X) = 0$.

To show that $f(K) = 0$, consider the function $f(P)$ for a variable point P on line $AKXZ$. We claim that

$$f(P) = \frac{PX \cdot PK}{AX \cdot AK} f(A),$$

after which the result follows from taking $P = K$. The LHS and RHS are quadratic functions that agree at A and X. It remains to check that they agree at one more point – we will take $P = Z$. We compute

$$\begin{aligned}
f(Z) &= \frac{BZ \cdot CZ}{R^2} = \frac{a^2b^2c^2}{(b^2+c^2)^2} \cdot \frac{1}{R^2} \\
f(A) &= -\frac{d(A, BC)}{d(O, BC)} \cdot \frac{d(A, UV)}{d(O, UV)} = \frac{d(A, BC)}{d(O, BC)} = \frac{bc}{2R^2 \cos A} \\
\frac{ZK}{AK} &= \frac{b^2+c^2}{a^2} \\
\frac{ZX}{AX} &= \frac{[XBC]}{[XABC]} = \frac{XB \cdot XC \sin(2A)}{XB \cdot XA \sin(A) + XA \cdot XC \sin(A)} = \frac{2bc \cos A}{b^2+c^2},
\end{aligned}$$

and the desired result follows.

Solution 3: We prove the following main claim.

Claim. Let P be a point on AX, and let the conic through B, C, O, X, P meet (ABC) again at U', V'. Then $U'V'$ passes through a fixed point, and the map $P \mapsto U'V' \cap AX$ is projective.

Proof. Take a projective transformation sending B, C to the circle points. Then the new problem is the following: We have points A, X, O, a fixed circle Ω passing through A, and a point P on AX. Let (POX) meet Ω at $U'V'$. Then $U'V'$ passes through a fixed point and $P \mapsto U'V' \cap AX$ is projective. First, if we take two points P_1, P_2, by radical axis on (P_1OX), (P_2OX), and Ω, $U_1'V_1'$, $U_2'V_2'$, and OX concur. Thus, $U'V'$ meets OX at a fixed point. Let this point be Q. Now let O_1 be the center of Ω and O_P the center of (POX). O_P moves linearly w.r.t. P, so $P \mapsto O_1O_P$ is projective. Then $U'V'$ is the line through Q perpendicular to O_1O_P, and hence also moves projectively. Finally, intersecting $U'V'$ with AX is projective, so the entire map $P \mapsto U'V' \cap AX$ is projective, as desired. □

We now return to the original problem. Let the tangents to (ABC) at B, C meet at T. It is well-known that A, X, K, T are collinear, and B, O, C, X, T are concyclic. We now consider the following cases:

- $\boldsymbol{P = AX \cap BC}$: The conic $(BCOXP)$ degenerates to the two lines OX, BC, so $U'V' = OX$, and $U'V' \cap AX = X$. Thus, $AX \cap BC \mapsto X$.
- $\boldsymbol{P = T}$: The conic $(BCOXT)$ is just the circle (BOC). Then (BOC) meets (ABC) again at the circle points, so $U'V'$ is the line at infinity, and $T \mapsto (AX)_\infty$.
- $\boldsymbol{P = A}$: In this case, $A = U'$ or V', so $U'V' \cap AX = A$ and $A \mapsto A$.

From the first two cases, we see that the fixed point $U'V'$ passes through is $(OX)_\infty$ i.e. $U'V'$ is always parallel to OX. Now note that the cross-ratio $(A, AX \cap BC; K, T) = -1$, and hence if $K \mapsto f(K)$, we have $(A, X; f(K), (AX)_\infty) = -1$. Thus, $f(K)$ must be the midpoint of AX, and UV is the line through $f(K)$ parallel to OX i.e. the perpendicular bisector of AX.