HMMT November 2024

November 09, 2024

General Round

1. Six consecutive positive integers are written on slips of paper. The slips are then handed out to Ethan, Jacob, and Karthik, such that each of them receives two slips. The product of Ethan's numbers is 20, and the product of Jacob's numbers is 24. Compute the product of Karthik's numbers.

   Proposed by: Luke Robitaille

   Answer: $\boxed{42}$

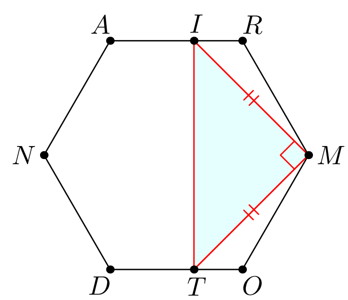
   Solution: Each person's numbers differ by at most 5, so Alice must have 4 and 5. Bob could have 4 and 6 or 3 and 8. Since Alice already has 4, Bob cannot have 4 and 6. So, Bob has 3 and 8. Then the six numbers must be 3 through 8, so Charlie has 6 and 7, multiplying to $\boxed{42}$.

2. Let $RANDOM$ be a regular hexagon with side length 1. Points I and T lie on segments $\overline{RA}$ and $\overline{DO}$, respectively, such that $MI = MT$ and $\angle TMI = 90^\circ$. Compute the area of triangle MIT.

   Proposed by: Linus Yifeng Tang

   Answer: $\boxed{\frac{3}{4} = 0.75}$

   Solution:



   By symmetry, IT must be perpendicular to RA and DO. Therefore, the length of IT is the height of the hexagon, which is $\sqrt{3}$. So, the area of triangle MIT is $\frac{1}{4}IT^2 = \boxed{\frac{3}{4}}$.

3. Suppose that a, b, and c are *distinct* positive integers such that $a^b b^c = a^c$. Across all possible values of a, b, and c, compute the minimum value of $a + b + c$.

   Proposed by: Derek Liu

   Answer: $\boxed{13}$

   Solution: We claim that $(8, 2, 3)$ is the desired solution.

   Observe that $a^{c-b} = b^c$, so clearly $a \neq 1$ and $b < a$. Furthermore, a and b must be distinct powers of the same integer.

   If a and b were powers of an integer $n > 2$, then we would have $a + b + c \geq 3^2 + 3 + 1 = 13$. Thus, we only need to consider when they are powers of 2.

   If $(a, b) = (4, 2)$ then $(c - b) = \frac{c}{2}$, so $c = 4$, which makes the values not distinct.

   If $(a, b) = (8, 2)$ we get our aforementioned solution.

   Any other (a, b) sum to at least 12, in which case $a + b + c \geq 13$.

   Thus $\boxed{13}$ is minimal.

4. Compute the number of ways to pick a 3-element subset of

$$\{10^1+1, 10^2+1, 10^3+1, 10^4+1, 10^5+1, 10^6+1, 10^7+1\}$$

such that the product of the 3 numbers in the subset has no digits besides 0 and 1 when written in base 10.

Proposed by: Albert Wang

Answer: $\boxed{26}$

Solution: Given a subset $\{10^a+1, 10^b+1, 10^c+1\}$, we can directly expand the product of its elements:

$$(10^a+1)(10^b+1)(10^c+1) = 10^{a+b+c} + 10^{b+c} + 10^{a+c} + 10^{a+b} + 10^a + 10^b + 10^c + 1.$$

In order for all digits to be 0 or 1, all 7 numbers $a+b+c, b+c, a+c, a+b, a, b, c$ should be distinct and nonzero, with the latter being guaranteed.

Without loss of generality, we can assume $c > b > a$. Then,

$$a+b+c > b+c > a+c > \max(a+b, c)$$

and

$$\min(a+b, c) > b > a,$$

so the only two numbers that could be the same are $a+b$ and c.

There are 9 triples (a, b, c) where $1 \le a < b < c \le 7$ and $a+b = c$, namely $(1,2,3)$, $(1,3,4)$, $(1,4,5)$, $(1,5,6)$, $(1,6,7)$, $(2,3,5)$, $(2,4,6)$, $(2,5,7)$, and $(3,4,7)$. The remaining triples (a, b, c) all work, so the answer is $\binom{7}{3} - 9 = \boxed{26}$.

5. Let f be a function on nonnegative integers such that $f(0) = 0$ and

$$f(3n+2) = f(3n+1) = f(3n) + 1 = 3f(n) + 1$$

for all integers $n \ge 0$. Compute the sum of all nonnegative integers m such that $f(m) = 13$.

Proposed by: Carlos Rodriguez

Answer: $\boxed{156}$

Solution: Let $\underline{x}_k$ denote the number x in base k. Observe that if $f(\underline{x}_3) = \underline{y}_3$, then

$$f\left(\underline{x0}_3\right) = f(3x) = 3f(x) = \underline{y0}_3$$

and

$$f\left(\underline{x1}_3\right) = f\left(\underline{x2}_3\right) = 3f(x) + 1 = \underline{y1}_3.$$

Thus, $\underline{f(n)}_3$ is simply $\underline{n}_3$ with all 2's replaced with 1's. We also see that $13 = 111_3$. Thus, $f(m) = 13$ if and only if $m = \underline{abc}_3$ for digits $a, b, c \in \{1, 2\}$. Each of a, b, and c takes on each possible value exactly 4 times, so the sum is

$$(4 \cdot 1 + 4 \cdot 2)(3^2 + 3^1 + 3^0) = \boxed{156}.$$

6. A positive integer n is *stacked* if $2n$ has the same number of digits as n and the digits of $2n$ are multiples of the corresponding digits of n. For example, 1203 is stacked because $2 \times 1203 = 2406$, and $2, 4, 0, 6$ are multiples of $1, 2, 0, 3$, respectively. Compute the number of stacked integers less than 1000.

Proposed by: Srinivas Arun

Answer: $\boxed{135}$

Solution: We do casework on the number of digits of n.

One digit. There are 4 one-digit stacked integers: 1, 2, 3, 4.

Two digits. Suppose $n = \overline{ab}$ is a two-digit integer. If $a < 5$ and $b < 5$, then the digits of $2n$ are double the respective digits of n, so n is stacked; there are $4 \cdot 5 = 20$ such n. Otherwise, since $2n < 100$, we still must have $a < 5$, so $b \geq 5$. Then the last digit of $2n$ is $2b - 10$, so $b \mid 2b - 10$, which implies that $b = 5$. Then the first digit of $2n$ is $2a+1$, which a must divide, so $a = 1$. Thus, the only stacked n with $b \geq 5$ is 15. Adding that to the 20 stacked numbers with $b < 5$ gives us 21 two-digit stacked integers.

Three digits. Suppose $n = \overline{abc}$ is a three-digit integer. If a, b, and c are all less than 5, then the digits of $2n$ are double the respective digits of n, so n is stacked; there are $4 \cdot 5 \cdot 5 = 100$ such n. Otherwise, since $2n < 1000$, we must have $a < 5$. We now casework on which of b and c are at least 5.

- If $b \geq 5$ and $c \geq 5$, then the digits of $2n$ are $2a+1$, $2b-9$, and $2c-10$ in order. Thus, $a \mid 2a+1$, $b \mid 2b-9$, and $c \mid 2c-10$, which implies $a = 1$, $b = 9$, and $c = 5$. Thus 195 is the only stacked number in this case.
- If $c \geq 5$ only, then $2n = 200a + 2\overline{bc}$ has first digit $2a$ and last two digits $2\overline{bc}$, so n is stacked if and only if $\overline{bc}$ to be stacked. Since $c \geq 5$, as proved before, the only such stacked $\overline{bc}$ is 15, so we get 4 stacked numbers in this case: 115, 215, 315, and 415.
- If $b \geq 5$ only, then $2n$ has last digit $2c$ and first two digits $2\overline{ab}$, so n is stacked if and only if $\overline{ab}$ to be stacked. As $b \geq 5$, similar to the previous case, the only such stacked $\overline{ab}$ is $\overline{ab} = 15$, so we get 5 stacked numbers in this case: 150, 151, 152, 153, and 154.

Summing over all cases, there are $100 + 1 + 4 + 5 = 110$ three-digit stacked integers.

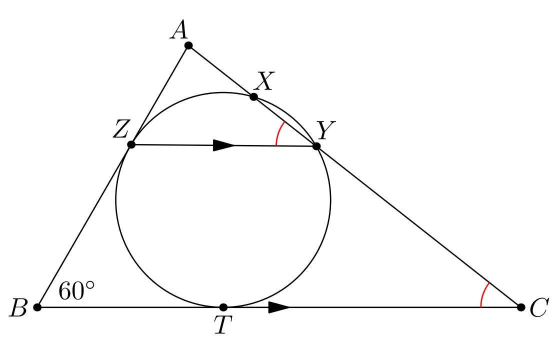
Our final answer is $4 + 21 + 110 = \boxed{135}$.

7. Let triangle ABC have $AB = 5, BC = 8$, and $\angle ABC = 60^\circ$. A circle ω tangent to segments $\overline{AB}$ and $\overline{BC}$ intersects segment $\overline{CA}$ at points X and Y such that points C, Y, X, and A lie along $\overline{CA}$ in this order. If ω is tangent to $\overline{AB}$ at point Z and $ZY \parallel BC$, compute the radius of ω.

Proposed by: Ethan Liu

Answer: $\boxed{\frac{40}{13\sqrt{3}} = \frac{40\sqrt{3}}{39}}$

Solution:



Let ω tangent to BC at T. Observe that $BT = BZ$ and $\angle ABC = 60^\circ$, so $\triangle TBZ$ is equilateral. Moreover, the tangent to ω at T is parallel to BC, so $TY = TZ$. Combining this with $\angle TZY = \angle ZTB = 60^\circ$, it follows that $\triangle TYZ$ is equilateral as well.

Now, let $BT = BZ = TZ = TY = YZ = x$. Then, $AZ = 5 - x$. Thus, similar triangles AYZ and ABC gives

$$\frac{AZ}{AB} = \frac{YZ}{BC} \implies \frac{5-x}{5} = \frac{x}{8}.$$

Solving this equation gives $40 - 8x = 5x$, or $x = \frac{40}{13}$. Finally, since $\triangle TYZ$ is an equilateral triangle inscribed in ω, the radius of ω is $\frac{x}{\sqrt{3}} = \boxed{\frac{40}{13\sqrt{3}}}$.

8. Let

$$f(x) = \left|\left|\cdots\left|\left|\left||x|-1\right|-2\right|-3\right|-\cdots\right|-10\right|.$$

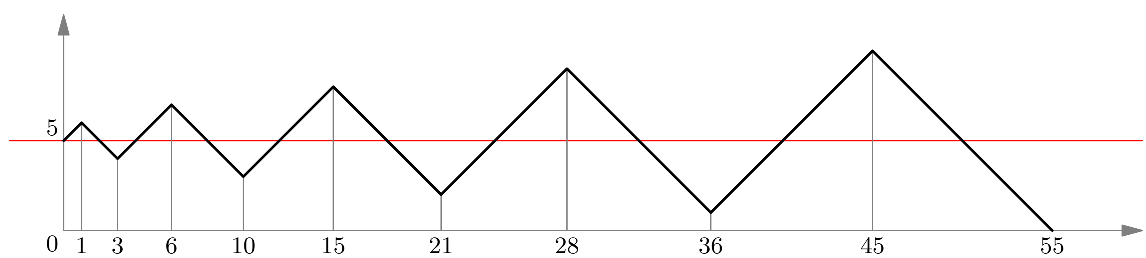
Compute $f(1) + f(2) + \cdots + f(54) + f(55)$.

Proposed by: Benjamin Shimabukuro

Answer: $\boxed{285}$

Solution: Let T_k denote the k-th triangular number $1 + 2 + \cdots + k$.

For any integer i, the function $g_i(x) = |x - i|$ is a piecewise linear function with slopes ± 1. As $f(x) = g_{10}(\cdots(g_1(g_0(x)))\cdots)$, it is also piecewise linear with slopes ± 1. As $g_i(x)$ has a cusp only where it evaluates to 0, the cusps of f occur precisely where $g_k(\cdots(g_1(g_0(x)))\cdots) = 0$ for some integer $0 \le k \le 10$. Then, $g_{k-1}(\cdots(g_0(x))\cdots) = \pm k$, and since it is positive, it equals k. Similarly, $g_{k-2}(\cdots(g_0(x))\cdots) = (k-1) \pm k$, so it must be $k + (k-1)$. Continuing this argument, we see that $|x| = k + (k-1) + \cdots + 1 = T_k$, so the cusps occur precisely when $|x|$ is a triangular between 0 and 55, inclusive. As $f(0) = 5$ and $f(1) = 6$, the graph of $f(x)$ on $0 \le x \le 55$ looks as follows:



Now observe that for $1 \le k \le 9$, the $(k+1)$ values $g(T_k)$, $g(T_k + 1)$, $\ldots$, $g(T_{k+1} - 1)$ are $5 - \frac{k}{2}$ through $5 + \frac{k}{2}$ if k is even, and $5.5 + \frac{k}{2}$ through $5.5 - \frac{k}{2}$ if k is odd. Thus they average to 5 if k is even and 5.5 if k is odd. As $f(55) = 0$, the desired sum is

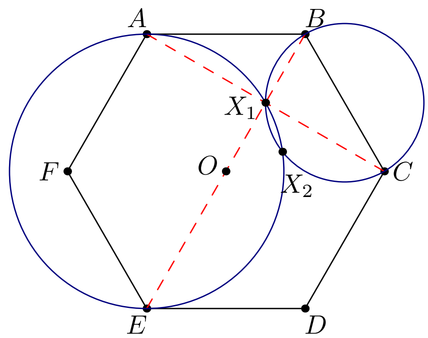
$$5 \cdot (3 + 5 + 7 + 9) + 5.5 \cdot (2 + 4 + 6 + 8 + 10) = \boxed{285}.$$

9. Let $ABCDEF$ be a regular hexagon with center O and side length 1. Point X is placed in the interior of the hexagon such that $\angle BXC = \angle AXE = 90^\circ$. Compute all possible values of OX.

Proposed by: Ethan Liu, Isabella Zhu, Pitchayut Saengrungkongka

Answer: $\boxed{\frac{1}{2}, \frac{\sqrt{7}}{7}}$

Solution 1:



Point X is the intersection of circles with diameter AE and BC. Thus, there are two possible intersection points. Since $AC \perp BE$, the first point, X_1, is the intersection of AC and BE, from which we can see $OX_1 = \boxed{\frac{1}{2}}$ as our first answer. Let X_2 be the other intersection point.

Let M be the midpoint of BC and N be the midpoint of AE. Then $MX_2 = NO = \frac{1}{2}$ and $MO = NX_2 = \frac{\sqrt{3}}{2}$, so OX_2MN is an isosceles trapezoid. By law of cosine on $\triangle OMN$, we have

$$\begin{aligned} MN &= \sqrt{OM^2 + ON^2 - 2 \cdot OM \cdot ON \cos 150^\circ} \\ &= \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}} = \frac{\sqrt{7}}{2}. \end{aligned}$$

Moreover, by Ptolemy's theorem,

$$OX_2 \cdot MN = MO^2 - NO^2 = \frac{1}{2}.$$

Combining the previous two equations gives $OX_2 = \boxed{\frac{\sqrt{7}}{7}}$.

Solution 2: Recall the first paragraph of the previous solution that $X_1 = AC \cap BE$ is the first point. Thus, the second point X_2 is the Miquel point of cyclic quadrilateral $ACBE$.

By a well-known property of Miquel point, if $Y = AB \cap CE$, then Y and X_2 are inverses with respect to the circumcircle of $ABCDEF$. Thus, $OX_2 \cdot OY = 1$.

One can compute OY as follows: from triangle BCY, we get that $BY = BC/\sin 30^\circ = 2$. Thus, by power of point,

$$OY^2 - 1^2 = YB \cdot YA = 2 \cdot 3 = 6 \implies OY = \sqrt{7},$$

implying $OX_2 = \boxed{\frac{\sqrt{7}}{7}}$.

10. Let $S = \{1, 2, 3, \ldots, 64\}$. Compute the number of ways to partition S into 16 arithmetic sequences such that each arithmetic sequence has length 4 and common difference 1, 4, or 16.

Proposed by: Isabella Zhu

Answer: $\boxed{203}$

Solution: The key observation is the following:

Claim 1. No partition can contain all three common differences.

Proof. Indeed, suppose the sequences x, $x + 16$, $x + 32$, $x + 48$ and y, $y + 4$, $y + 8$, $y + 12$ are both present for some x and y in S. Without loss of generality, assume $y \leq 26$; otherwise, we can take our partition and replace each number n with $65 - n$, resulting in the sequence $53 - y$, $57 - y$, $61 - y$, $65 - y$ instead.

Note that $y \not\equiv x \bmod 4$, as otherwise one of y, $y+4$, $y+8$, or $y+12$ would be equivalent to x modulo 16 and the two sequences would intersect.

Hence, there exists a number z strictly between y and $y+4$ which is equivalent to x modulo 4. The same argument above tells us z cannot be in a difference-4 sequence; it also cannot be in a difference-1 sequence, as such a sequence would contain either y or $y+4$. Thus z is in a difference-16 sequence. Similarly, as $z+4$ lies between $y+4$ and $y+8$, and $z+8$ lies between $y+8$ and $y+12$, both $z+4$ and $z+8$ are in difference-4 sequences.

Since we assumed $y \leq 26$, we know $y+32 \in S$. Note that $y+20$ cannot be part of a difference-16 sequence, as such a sequence would also contain $y+4$. Furthermore, $y+20$ lies between $z+16$ and $z+20$, both of which are in difference-4 sequences; hence, $y+20$ cannot be part of a difference-1 sequence. Thus $y+20$ is in a difference-4 sequence. This sequence must contain either $y+16$ or both $y+28$ and $y+32$.

If the sequence contains $y+16$, then since $z+12$ lies strictly between $y+12$ and $y+16$, the same argument as before tells us $z+12$ is in a difference-16 sequence. If the sequence contains $y+28$ and $y+32$, then $z+28$ lies strictly between the two, so $z+28$ is in a difference-16 sequence; this sequence contains $z+12$. In either case, $z+12$ is in a difference-16 sequence.

Now, we know z, $z+4$, $z+8$, and $z+12$ are all in difference-16 sequences. These sequences contain all 16 numbers in the same residue class as z modulo 4. Any difference-1 sequence would have to contain a value in this residue class; thus, no difference-1 sequences can be present. $\square$

We casework on which types of sequences are present.

Case 1: Only sequences of common difference 1 and 16 appear.

Observe that each sequence of common difference 1 has one number of each residue class modulo 4, while each sequence of common difference 16 has four numbers in the same residue class. Since S has an equal number of elements in each residue class, there must be an equal number of difference-16 sequences in each residue class, so the number of difference-16 sequences is a multiple of 4. Say there are $4x$ of them.

Then, among the numbers 1 through 16, there are $4x$ of them that lie in difference-16 sequences, so the remaining $16-4x$ lie in $4-x$ difference-1 sequences.

Conversely, if we are given how the numbers from 1 through 16 are split between difference-1 and difference-16 sequences, we can uniquely recover the whole partition on S. Indeed, the difference-16 sequences are fixed, which in turn fixes the difference-1 sequences.

Thus, the number of sequences in this case is the number of ordered partitions of 16 into $4x$ 1s and $4-x$ 4s, which is $\binom{4+3x}{4-x}$. Summing over all x, the total for this case is

$$\binom{16}{0} + \binom{13}{1} + \binom{10}{2} + \binom{7}{3} + \binom{4}{4} = 95.$$

Case 2: Only sequences of common difference 4 and 16 appear.

Within the multiples of 4, any difference-4 and difference-16 sequence intersect, so the 16 multiples of 4 must be covered with either four difference-4 sequences or four difference-16 sequences. The same goes for each residue class mod 4, and we can make each choice independently. Thus the number of partitions in this case is $2^4 = 16$.

Case 3: Only sequences of common difference 1 and 4 appear.

Observe that if x and $x+4$ are in a difference-4 sequence, then $x+1$, $x+2$, and $x+3$ must also be in difference-4 sequences, so the difference-4 sequences form contiguous blocks of length $4 \cdot 4 = 16$. The difference-1 sequences are themselves contiguous blocks of length 4, so the number of sequences in this case is the number of ordered partitions of 64 into 4s and 16s. This is the same as the number of ordered partitions of 16 into 1s and 4s, which we calculated in Case 1; there are 95 of them.

Summing over all cases, we get $95+16+95 = 206$. However, we overcount any partition with only one type of sequence, of which there are three (one for each type). Thus, the answer is $206 - 3 = \boxed{203}$.