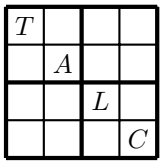
HMMT November 2024

November 09, 2024

Theme Round

1. Compute the number of ways to fill each of the 12 empty cells in the grid on the right with one of T, A, L, or C such that each of the four rows, columns, and bolded 2×2 square regions contains each letter exactly once.



2. Paul is in the desert and has a pile of gypsum crystals. No matter how he divides the pile into two nonempty piles, at least one of the resulting piles has a number of crystals that, when written in base 10, has a sum of digits at least 7. Given that Paul's initial pile has at least two crystals, compute the smallest possible number of crystals in the initial pile.

3. Points K, A, L, C, I, T, E are such that triangles CAT and ELK are equilateral, share a center I, and points E, L, K lie on sides $\overline{CA}, \overline{AT}, \overline{TC}$ respectively. If the area of triangle CAT is double the area of triangle ELK and $CI = 2$, compute the minimum possible value of CK.

4. Compute
$$\sum_{i=1}^{4} \sum_{t=1}^{4} \sum_{e=1}^{4} \left\lfloor \frac{ite}{5} \right\rfloor.$$

5. Alf, the alien from the 1980s TV show, has a big appetite for the mineral apatite. However, he's currently on a diet, so for each integer $k \geq 1$, he can eat exactly k pieces of apatite on day k. Additionally, if he eats apatite on day k, he cannot eat on any of days $k+1$, $k+2$, $\ldots$, $2k-1$. Compute the maximum total number of pieces of apatite Alf could eat over days $1, 2, \ldots, 99, 100$.

6. Let $FELDSPAR$ be a regular octagon, and let I be a point in its interior such that $\angle FIL = \angle LID = \angle DIS = \angle SIA$. Compute $\angle IAR$ in degrees.

7. Jasper and Rose are playing a game. Twenty-six 32-ounce jugs are in a line, labeled Quart A through Quart Z from left to right. All twenty-six jugs are initially full. Jasper and Rose take turns making one of the following two moves:

   - remove a positive integer number of ounces (possibly all) from the leftmost nonempty jug, or
   - remove an *equal* positive integer number of ounces from the two leftmost nonempty jugs, possibly emptying them. Neither player may remove more ounces from a jug than it currently contains.

   Jasper plays first. A player's score is the number of ounces they take from Quart Z. If both players play to maximize their score, compute the maximum score that Jasper can guarantee.

8. For all positive integers r and s, let $\text{Top}(r, s)$ denote the top number (i.e., numerator) when $\frac{r}{s}$ is written in simplified form. For instance, $\text{Top}(20, 24) = 5$. Compute the number of ordered pairs of positive integers (a, z) such that $200 \leq a \leq 300$ and $\text{Top}(a, z) = \text{Top}(z, a-1)$.

9. Compute the number of ways to color each cell of an 18×18 square grid either ruby or sapphire such that each contiguous 3×3 subgrid has exactly 1 ruby cell.

10. Isabella the geologist discovers a diamond deep underground using an X-ray machine. The diamond has the shape of a convex cyclic pentagon $PABCD$ with $AD \parallel BC$. Soon after the discovery, her X-ray breaks, and she only recovers partial information about its dimensions. She knows that $AD = 70$, $BC = 55$, $PA : PD = 3 : 4$, and $PB : PC = 5 : 6$. Compute PB.