HMMT February 2025

February 15, 2025

Algebra and Number Theory Round

1. Compute the sum of the positive divisors (including 1) of $9!$ that have units digit 1.

2. Mark writes the expression $\sqrt{\underline{abcd}}$ on the board, where $\underline{abcd}$ is a four-digit number and $a \neq 0$. Derek, a toddler, decides to move the a, changing Mark's expression to $a\sqrt{\underline{bcd}}$. Surprisingly, these two expressions are equal. Compute the only possible four-digit number $\underline{abcd}$.

3. Given that x, y, and z are positive real numbers such that
$$x^{\log_2(yz)} = 2^8 \cdot 3^4, \quad y^{\log_2(zx)} = 2^9 \cdot 3^6, \quad \text{and} \quad z^{\log_2(xy)} = 2^5 \cdot 3^{10},$$
compute the smallest possible value of xyz.

4. Let $\lfloor z \rfloor$ denote the greatest integer less than or equal to z. Compute
$$\sum_{j=-1000}^{1000} \left\lfloor \frac{2025}{j+0.5} \right\rfloor.$$

5. Let $\mathcal{S}$ be the set of all nonconstant monic polynomials P with integer coefficients satisfying $P\left(\sqrt{3}+\sqrt{2}\right) = P\left(\sqrt{3}-\sqrt{2}\right)$. If Q is an element of $\mathcal{S}$ with minimal degree, compute the only possible value of $Q(10)-Q(0)$.

6. Let r be the remainder when $2017^{2025!}-1$ is divided by $2025!$. Compute $\frac{r}{2025!}$. (Note that 2017 is prime.)

7. There exists a unique triple (a,b,c) of positive real numbers that satisfies the equations
$$2(a^2+1) = 3(b^2+1) = 4(c^2+1) \quad \text{and} \quad ab+bc+ca = 1.$$
Compute $a+b+c$.

8. Define $\operatorname{sgn}(x)$ to be 1 when x is positive, -1 when x is negative, and 0 when x is 0. Compute
$$\sum_{n=1}^{\infty} \frac{\operatorname{sgn}(\sin(2^n))}{2^n}.$$
(The arguments to sin are in radians.)

9. Let f be the unique polynomial of degree at most 2026 such that for all $n \in \{1,2,3,\ldots,2027\}$,
$$f(n) = \begin{cases} 1 & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise.} \end{cases}$$
Suppose that $\frac{a}{b}$ is the coefficient of x^{2025} in f, where a and b are integers such that $\gcd(a,b)=1$. Compute the unique integer r between 0 and 2026 (inclusive) such that $a-rb$ is divisible by 2027. (Note that 2027 is prime.)

10. Let a, b, and c be pairwise distinct complex numbers such that
$$a^2 = b+6, \quad b^2 = c+6, \quad \text{and} \quad c^2 = a+6.$$
Compute the two possible values of $a+b+c$.