HMMT February 2025

February 15, 2025

Algebra and Number Theory Round

1. Compute the sum of the positive divisors (including 1) of $9!$ that have units digit 1.

   Proposed by: Jackson Dryg

   Answer: $\boxed{103}$

   Solution: The prime factorization of $9!$ is $2^7 \cdot 3^4 \cdot 5 \cdot 7$. Every divisor of $9!$ has prime factorization $2^a \cdot 3^b \cdot 5^c \cdot 7^d$, where $0 \le a \le 7$, $0 \le b \le 4$, $0 \le c \le 1$, and $0 \le d \le 1$. If the divisor has units digit 1, it cannot be divisible by 2 or 5, so $a = c = 0$.

   Now take cases on the value of d:

   - If $d = 0$, then the divisor is 3^b for some $0 \le b \le 4$. The possible divisors are 1, 3, 9, 27, and 81, of which 1 and 81 work.
   - If $d = 1$, then the divisor is $3^b \cdot 7$ for some $0 \le b \le 4$. The possible divisors are then 7, $3 \cdot 7$, $9 \cdot 7$, $27 \cdot 7$, and $81 \cdot 7$. Of these, only $3 \cdot 7 = 21$ works.

   The answer is $1 + 21 + 81 = \boxed{103}$.

2. Mark writes the expression $\sqrt{\underline{abcd}}$ on the board, where $\underline{abcd}$ is a four-digit number and $a \neq 0$. Derek, a toddler, decides to move the a, changing Mark's expression to $a\sqrt{\underline{bcd}}$. Surprisingly, these two expressions are equal. Compute the only possible four-digit number $\underline{abcd}$.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{3375}$

   Solution: Let $x = \underline{bcd}$. Then, we rewrite the given condition $\sqrt{\underline{abcd}} = a\sqrt{\underline{bcd}}$ as

   $$1000a + x = a^2 x,$$

   which simplifies as

   $$(a^2 - 1)x = 1000a.$$

   In particular, $a^2 - 1$ divides $1000a$. Since $\gcd(a^2 - 1, a) = 1$, it follows that $a^2 - 1 \mid 1000$. The only $a \in \{1, 2, \ldots, 9\}$ that satisfies this is $a = 3$. Then $8x = 3000$, so $x = 375$. Thus $\underline{abcd} = \boxed{3375}$.

3. Given that x, y, and z are positive real numbers such that

   $$x^{\log_2(yz)} = 2^8 \cdot 3^4, \quad y^{\log_2(zx)} = 2^9 \cdot 3^6, \quad \text{and} \quad z^{\log_2(xy)} = 2^5 \cdot 3^{10},$$

   compute the smallest possible value of xyz.

   Proposed by: Derek Liu

   Answer: $\boxed{\frac{1}{576}}$

   Solution: Let $k = \log_2 3$ for brevity. Taking the base-2 log of each equation gives

   $$\begin{aligned}(\log_2 x)(\log_2 y + \log_2 z) &= 8 + 4k,\\ (\log_2 y)(\log_2 z + \log_2 x) &= 9 + 6k,\\ (\log_2 z)(\log_2 x + \log_2 y) &= 5 + 10k.\end{aligned}$$

Adding the first two equations and subtracting the third yields $2\log_2 x\,\log_2 y = 12$, so $\log_2 x\,\log_2 y = 6$. Similarly, we get

$$\begin{aligned}\log_2 x\,\log_2 y &= 6,\\ \log_2 y\,\log_2 z &= 3+6k,\\ \log_2 z\,\log_2 x &= 2+4k.\end{aligned}$$

Multiplying the first two equations and dividing by the third yields $(\log_2 y)^2 = 9$, so $\log_2 y = \pm 3$. Then, the first and last equations tell us $\log_2 x = \pm 2$ and $\log_2 z = \pm(1+2k)$, with all signs matching. Thus

$$\log_2 x + \log_2 y + \log_2 z = \pm(3+2+(1+2k)) = \pm(6+2k),$$

so

$$xyz = 2^{\pm(6+2k)} = 2^6\cdot 3^2 \quad \text{or} \quad 2^{-6}\cdot 3^{-2}.$$

Clearly, the smallest solution is $2^{-6}\cdot 3^{-2} = \boxed{\frac{1}{576}}$.

4. Let $\lfloor z\rfloor$ denote the greatest integer less than or equal to z. Compute

$$\sum_{j=-1000}^{1000}\left\lfloor \frac{2025}{j+0.5}\right\rfloor.$$

Proposed by: Linus Yifeng Tang

Answer: $\boxed{-984}$

Solution: The key idea is to pair up the terms $\left\lfloor\frac{2025}{-x}\right\rfloor$ and $\left\lfloor\frac{2025}{x}\right\rfloor$. There are 1000 such pairs and one lone term, $\left\lfloor\frac{2025}{1000.5}\right\rfloor = 2$. Thus,

$$\sum_{j=-1000}^{1000}\left\lfloor \frac{2025}{j+0.5}\right\rfloor = 2 + \sum_{x\in\{0.5,1.5,\ldots,999.5\}}\left(\left\lfloor\frac{2025}{x}\right\rfloor + \left\lfloor\frac{2025}{-x}\right\rfloor\right).$$

We note that

$$\lfloor a\rfloor + \lfloor -a\rfloor = \begin{cases}0 & \text{if } a \text{ is an integer.}\\ -1 & \text{otherwise.}\end{cases}$$

Therefore,

$$\left\lfloor\frac{2025}{x}\right\rfloor + \left\lfloor\frac{2025}{-x}\right\rfloor = \begin{cases}0 & \text{if } 2x \text{ divides } 4050\\ -1 & \text{otherwise.}\end{cases}$$

As x ranges in the set $\{0.5, 1.5, 2.5, \ldots, 999.5\}$, $2x$ ranges in the set $\{1, 3, 5, \ldots, 1999\}$. This set includes all 15 odd divisors of 4050 except for 2025. Thus, there are 14 values of x for which $\left\lfloor\frac{2025}{x}\right\rfloor + \left\lfloor\frac{2025}{-x}\right\rfloor$ evaluates to 0, and the remaining $1000 - 14 = 986$ values of x make it evaluate to -1. Therefore,

$$\sum_{j=-1000}^{1000}\left\lfloor \frac{2025}{j+0.5}\right\rfloor = 2 + \sum_{x\in\{0.5,1.5,\ldots,999.5\}}\left(\left\lfloor\frac{2025}{x}\right\rfloor + \left\lfloor\frac{2025}{-x}\right\rfloor\right) = 2 + 986\cdot(-1) = \boxed{-984}.$$

5. Let $\mathcal{S}$ be the set of all nonconstant monic polynomials P with integer coefficients satisfying $P\left(\sqrt{3}+\sqrt{2}\right) = P\left(\sqrt{3}-\sqrt{2}\right)$. If Q is an element of $\mathcal{S}$ with minimal degree, compute the only possible value of $Q(10)-Q(0)$.

   Proposed by: David Dong

   Answer: $\boxed{890}$

   Solution: First, note that the polynomial x^4-10x^2+1 has both $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ as roots. It suffices to check whether a polynomial of degree at most 3 belongs in $\mathcal{S}$. Suppose $f(x)=ax^3+bx^2+cx+d\in\mathcal{S}$. We compute

$$(\sqrt{3}+\sqrt{2})^3-(\sqrt{3}-\sqrt{2})^3 = 22\sqrt{2}$$
$$(\sqrt{3}+\sqrt{2})^2-(\sqrt{3}-\sqrt{2})^2 = 4\sqrt{6}$$
$$(\sqrt{3}+\sqrt{2})^1-(\sqrt{3}-\sqrt{2})^1 = 2\sqrt{2},$$

   so we get that
$$f(\sqrt{3}+\sqrt{2})-f(\sqrt{3}-\sqrt{2}) = (22\sqrt{2})a+(4\sqrt{6})b+(2\sqrt{2})c.$$

   By resolving linear dependencies, it's clear that $b=0$ and $c=-11a$. It follows that if f is not the zero polynomial, it must be cubic. It is then clear that $f(x)=x^3-11x+d$ has minimal degree in $\mathcal{S}$, and thus $Q(10)-Q(0)=f(10)-f(0)=\boxed{890}$.

6. Let r be the remainder when $2017^{2025!}-1$ is divided by $2025!$. Compute $\frac{r}{2025!}$. (Note that 2017 is prime.)

   Proposed by: Srinivas Arun

   Answer: $\boxed{\frac{1311}{2017}}$

   Solution: Let $N=2017^{2025!}$. Let p be a prime dividing $2025!$ other than 2017. Let p^k be the largest power of p dividing $2025!$. Clearly, $\varphi(p^k)=(p-1)p^{k-1}$ divides $2025!$ and $\gcd(2017,p^k)=1$, so by Euler's Totient Theorem,
$$N\equiv 1 \pmod{p^k}.$$

   Repeating for all such primes p, we obtain
$$N\equiv 1 \pmod{2025!/2017}.$$

   Therefore, $\frac{2025!}{2017}\mid N-1$, so $r=\frac{2025!}{2017}s$ for some $0\le s<2017$. Also, since $N\equiv 0 \pmod{2017}$, we have $r=\frac{2025!}{2017}s\equiv -1 \pmod{2017}$.

   By Wilson's,
$$\frac{2025!}{2017} = 2016!(2018)(2019)\dots(2025)\equiv -8!\equiv 20 \pmod{2017}.$$

   Therefore, s is negative the inverse of $20 \pmod{2017}$, which is 1311. Our answer is
$$\frac{r}{2025!}=\frac{(2025!/2017)(1311)}{2025!}=\boxed{\frac{1311}{2017}}.$$

7. There exists a unique triple (a,b,c) of positive real numbers that satisfies the equations
$$2(a^2+1)=3(b^2+1)=4(c^2+1) \quad \text{and} \quad ab+bc+ca=1.$$

   Compute $a+b+c$.

Proposed by: David Wei

Answer: $\boxed{\frac{9\sqrt{23}}{23} = \frac{9}{\sqrt{23}}}$

Solution 1: The crux of this problem is to apply the trigonometric substitutions $a = \cot\alpha$, $b = \cot\beta$, and $c = \cot\gamma$, with $0 < \alpha, \beta, \gamma < \pi/2$. Then, the given equations translate to

$$\frac{2}{\sin^2\alpha} = \frac{3}{\sin^2\beta} = \frac{4}{\sin^2\gamma} \qquad \text{and} \qquad \cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1.$$

From the second equation, we get

$$\cot\gamma = \frac{1 - \cot\alpha\cot\beta}{\cot\alpha + \cot\beta} = -\cot(\alpha+\beta).$$

Since α, β, and γ all between 0 and $\pi/2$, we discover that

$$\alpha + \beta + \gamma = \pi.$$

Let $\triangle ABC$ be the (acute) triangle with side lengths $BC = \sqrt{2}$, $CA = \sqrt{3}$, and $AB = \sqrt{4}$. By Law of Sines, setting $\alpha = \angle A$, $\beta = \angle B$, and $\gamma = \angle C$ will satisfy both equations. Thus, Law of Cosines gives

$$\cos\alpha = \frac{3+4-2}{2\cdot\sqrt{3}\cdot\sqrt{4}} = \frac{5}{\sqrt{48}} \implies a = \cot\alpha = \frac{5}{\sqrt{23}}$$

Similar calculations give $b = \frac{3}{\sqrt{23}}$ and $c = \frac{1}{\sqrt{23}}$, so the answer is $a + b + c = \boxed{\frac{9}{\sqrt{23}}}$.

Solution 2: Let $2(a^2+1) = 3(b^2+1) = 4(c^2+1) = x$. Then, since $ab + bc + ca = 1$, we have the following system of equations:

$$\begin{aligned}
(a+b)(c+a) &= a^2 + ab + bc + ca = a^2 + 1 = x/2 \\
(b+c)(a+b) &= b^2 + ab + bc + ca = b^2 + 1 = x/3 \\
(c+a)(b+c) &= c^2 + ab + bc + ca = c^2 + 1 = x/4.
\end{aligned}$$

Taking advantage of symmetry, we discover that

$$a + b = \sqrt{\frac{2x}{3}}, \quad b + c = \sqrt{\frac{x}{6}}, \quad \text{and} \quad c + a = \sqrt{\frac{3x}{8}}.$$

To solve for x, notice that

$$\begin{aligned}
2 &= 2(ab + bc + ca) \\
&= (a+b)^2 + (b+c)^2 + (c+a)^2 - 2(a^2+b^2+c^2) \\
&= \frac{2x}{3} + \frac{x}{6} + \frac{3x}{8} - 2\left(\frac{x}{2} - 1 + \frac{x}{3} - 1 + \frac{x}{4} - 1\right) \\
&= -\frac{23x}{24} + 6,
\end{aligned}$$

so $x = \frac{96}{23}$. Therefore,

$$\begin{aligned}
a + b + c &= \frac{1}{2}\left(\sqrt{\frac{2x}{3}} + \sqrt{\frac{x}{6}} + \sqrt{\frac{3x}{8}}\right) \\
&= \frac{1}{2}\left(\frac{8+4+6}{\sqrt{23}}\right) = \boxed{\frac{9}{\sqrt{23}}}.
\end{aligned}$$

8. Define $\operatorname{sgn}(x)$ to be 1 when x is positive, -1 when x is negative, and 0 when x is 0. Compute

$$\sum_{n=1}^{\infty} \frac{\operatorname{sgn}(\sin(2^n))}{2^n}.$$

(The arguments to sin are in radians.)

Proposed by: Karthik Venkata Vedula

Answer: $\boxed{1 - \frac{2}{\pi}}$

Solution: Note that each of following is equivalent to the next.

- $\operatorname{sgn}(\sin(2^n)) = +1$.
- $0 < 2^n \bmod 2\pi < \pi$.
- $0 < \frac{2^n}{\pi} \bmod 2 < 1$.
- The nth digit after the decimal point in the binary representation of $\frac{1}{\pi}$ is 0.

Similarly, $\operatorname{sgn}(\sin(2^n)) = -1$ if and only if the n-th digit after the decimal point in the binary representation of $\frac{1}{\pi}$ is 1. In particular, if a_n is the n-th digit, then $\operatorname{sgn}(\sin(2^n)) = 1 - 2a_n$. Thus, the desired sum is

$$\sum_{n=1}^{\infty} \frac{\operatorname{sgn}(\sin(2^n))}{2^n} = \sum_{n=1}^{\infty} \frac{1 - 2a_n}{2^n} = \left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) - 2\left(\sum_{n=1}^{\infty} \frac{a_n}{2^n}\right) = \boxed{1 - \frac{2}{\pi}}.$$

9. Let f be the unique polynomial of degree at most 2026 such that for all $n \in \{1, 2, 3, \ldots, 2027\}$,

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\frac{a}{b}$ is the coefficient of x^{2025} in f, where a and b are integers such that $\gcd(a, b) = 1$. Compute the unique integer r between 0 and 2026 (inclusive) such that $a - rb$ is divisible by 2027. (Note that 2027 is prime.)

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{1037}$

Solution 1: Let $p = 2027$. We work in $\mathbb{F}_p$ for the entire solution. Recall the well-known fact that

$$\sum_{x \in \mathbb{F}_p} x^k = \begin{cases} -1 & \text{if } k > 0 \text{ and } p - 1 \mid k, \\ 0 & \text{otherwise,} \end{cases}$$

assuming $0^0 = 1$. In particular, for any polynomial $g(x) = b_0 + b_1 x + \cdots + b_n x^n$, we have

$$-\sum_{x \in \mathbb{F}_p} g(x) = b_{p-1} + b_{2(p-1)} + \cdots + b_{\lfloor n/(p-1) \rfloor (p-1)}.$$

We apply this fact on $g(x) = xf(x)$. As $\deg xf(x) \le p$, the right hand side is simply the coefficient of x^{2025}, which is what we want. Hence, the answer is

$$-\sum_{x \in \mathbb{F}_p} xf(x) = -(1^2 + 2^2 + \cdots + 45^2) = -\frac{45 \cdot 46 \cdot 91}{6} \equiv \boxed{1037} \pmod{2027}.$$

Solution 2: Again, let $p = 2027$ and work in $\mathbb{F}_p$. By the Lagrange Interpolation formula, we get that

$$f(x) = \sum_{i \in \mathbb{F}_p} f(i) \prod_{j \neq i} \frac{x-j}{i-j}.$$

We now simplify the polynomial in the product sign on the right-hand side. First, recall the identity

$$\prod_{j \in \mathbb{F}_p} (x-j) = x^p - x = (x-i)^p - (x-i).$$

The denominator $\prod_{j \neq i}(i-j)$ becomes $(p-1)! = -1$ by Wilson's. Thus, we get that

$$\prod_{j \neq i} \frac{x-j}{i-j} = -\frac{(x-i)^p - (x-i)}{x-i} = -(x-i)^{p-1} + 1.$$

The coefficient of x^{p-2} in the above expression is $-i$. Therefore, the first equation gives that the coefficient of x^{p-2} in $f(x)$ is

$$\sum_{i \in \mathbb{F}_p} -i f(i) = -(1^2 + 2^2 + \cdots + 45^2) = -\frac{45 \cdot 46 \cdot 91}{6} \equiv \boxed{1037} \pmod{2027}.$$

10. Let a, b, and c be pairwise distinct complex numbers such that

$$a^2 = b + 6, \quad b^2 = c + 6, \quad \text{and} \quad c^2 = a + 6.$$

Compute the two possible values of $a + b + c$.

Proposed by: Vasawat Rawangwong

Answer: $\boxed{\frac{-1+\sqrt{17}}{2}, \frac{-1-\sqrt{17}}{2}}$

Solution 1: Notice that any of a, b, or c being 3 or -2 implies $a = b = c$, which is invalid. Thus,

$$(a^2-9)(b^2-9)(c^2-9) = (b-3)(c-3)(a-3) \implies (a+3)(b+3)(c+3) = 1,$$
$$(a^2-4)(b^2-4)(c^2-4) = (b+2)(c+2)(a+2) \implies (a-2)(b-2)(c-2) = 1.$$

Therefore, 2 and -3 are roots of the polynomial $(x-a)(x-b)(x-c)+1$, and so there exists some t such that

$$(x-t)(x-2)(x+3) = (x-a)(x-b)(x-c) + 1.$$

Comparing coefficients gives $a + b + c = t - 1$ and $ab + bc + ca = -(t+6)$. We can then solve for t by noting $a^2 + b^2 + c^2 = (b+6) + (c+6) + (a+6) = a + b + c + 18$, so

$$ab + bc + ca = \frac{1}{2}((a+b+c)^2 - (a^2+b^2+c^2)) = \frac{1}{2}((a+b+c)^2 - (a+b+c+18)).$$

Hence,

$$-(t+6) = \frac{1}{2}((t-1)^2 - (t+17)) \implies t^2 - t - 4 = 0 \implies t = \frac{1 \pm \sqrt{17}}{2}.$$

Therefore $a + b + c = \boxed{\frac{-1\pm\sqrt{17}}{2}}$ are the two possible values of $a + b + c$.

Solution 2: Let $s = a + b + c$. Subtracting two adjacent equations gives $a^2 - b^2 = b - c$, or $(a-b)(a+b) = (b-c)$. Multiplying this and its cyclic variants gives

$$(a+b)(b+c)(c+a) = 1.$$

Now, we recall the identity

$$\begin{aligned}(a+b+c)^3 &= a^3+b^3+c^3+3(a+b)(b+c)(c+a)\\ \implies \quad s^3 &= a^3+b^3+c^3+3.\end{aligned}$$

To simplify $a^3+b^3+c^3$, we add a times the first equation, b times the second, and c times the third to obtain

$$\begin{aligned}a^3+b^3+c^3 &= a(b+6)+b(c+6)+c(a+6)\\ &= (ab+bc+ca)+6s\\ &= \tfrac{1}{2}\Big((a+b+c)^2-(a^2+b^2+c^2)\Big)+6s\\ &= \tfrac{1}{2}s^2-\tfrac{1}{2}\Big((b+6)+(c+6)+(a+6)\Big)+6s\\ &= \tfrac{1}{2}s^2+\tfrac{11}{2}s-9.\end{aligned}$$

Therefore,

$$s^3=\tfrac{1}{2}s^2+\tfrac{11}{2}s-6 \implies \left(s-\tfrac{3}{2}\right)\left(s^2+s-4\right)=0.$$

At this point, the only reasonable guess is that $s=\frac{3}{2}$ is an extra solution, and the remaining two roots $s=\boxed{\frac{-1\pm\sqrt{17}}{2}}$ are the possible answers. We now justify this guess. Assume for sake of contradiction that $s=\frac{3}{2}$. Then,

$$\begin{aligned}a^2+b^2+c^2 &= (b+6)+(c+6)+(a+6)=\tfrac{39}{2}\\ ab+bc+ca &= \tfrac{1}{2}\left(\tfrac{9}{4}-\tfrac{39}{2}\right)=-\tfrac{69}{8}.\end{aligned}$$

Then, observe

$$\begin{aligned}abc &= (a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a)\\ &= -\tfrac{207}{16}-1=-\tfrac{223}{16}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(a+6)(b+6)(c+6) &= 216+36(a+b+c)+6(ab+bc+ca)+abc\\ &= 216+36\cdot\tfrac{3}{2}-6\cdot\tfrac{69}{8}-\tfrac{223}{16},\end{aligned}$$

which is a rational number of denominator 16. But $(a+6)(b+6)(c+6)=b^2c^2a^2=\left(-\frac{223}{16}\right)^2$ has denominator $16^2=256$, a contradiction. Thus $s=\frac{3}{2}$ is impossible. (It arises from $a=b=c=\frac{1}{2}$, which satisfies $(a+b)(b+c)(c+a)=1$ but not the given conditions.)

Solution 3: Subtracting any two adjacent equations gives $a^2-b^2=b-c$, which is equivalent to both $(a-b)(a+b)=(b-c)$ and $(a-b)(a+b+1)=(a-c)$. Multiplying each of these with its respective cyclic variants and canceling the $(a-b)(b-c)(c-a)$ factor (which is given to be nonzero), we get

$$(a+b)(b+c)(c+a)=1 \quad \text{and} \quad (a+b+1)(b+c+1)(c+a+1)=-1.$$

Expanding the latter equation and using the given equations gives the following result.

$$\begin{aligned}(a+b)(b+c)(c+a)+(a^2+b^2+c^2)+3(ab+bc+ca)+2(a+b+c)+1&=-1\\ 1+(b+6+c+6+a+6)+3(ab+bc+ca)+2(a+b+c)+1&=-1\\ 3(a+b+c)+3(ab+bc+ca)&=-21\\ a+b+c+ab+bc+ca&=-7.\end{aligned}$$

Let $s = a + b + c$. We can then solve for s by considering the following:

$$\begin{aligned} s^2 &= (a^2 + b^2 + c^2) + 2(ab + bc + ca) \\ &= (b + 6 + c + 6 + a + 6) + 2(-7 - a - b - c) \\ &= -s + 4, \end{aligned}$$

so $s = \boxed{\frac{-1 \pm \sqrt{17}}{2}}$.

Solution 4: Let $s = a + b + c$ and consider the polynomial

$$x + (x^2 - 6) + ((x^2 - 6)^2 - 6) - s = x^4 - 11x^2 + x + 24 - s.$$

This polynomial has roots a, b, and c. By Vieta's, the sum of all four roots is 0, so its fourth root must be $-s$. Using Vieta's again, we have $ab + bc + ca - sa - sb - sc = -11$. We can now solve for s.

$$\begin{aligned} ab + bc + ca - (a + b + c)^2 &= -11 \\ a^2 + b^2 + c^2 + ab + bc + ca &= 11 \\ \tfrac{1}{2}((a + b + c)^2 + (a^2 + b^2 + c^2)) &= 11 \\ (a + b + c)^2 + (b + 6 + c + 6 + a + 6) &= 22 \\ s^2 + s - 4 = 0 \implies s &= \boxed{\frac{-1 \pm \sqrt{17}}{2}}. \end{aligned}$$

Remark. Another way to finish using this approach is to substitute $-s$ directly into $x^4 - 11x^2 + x + 24 - s = 0$ to get $(s - 3)(s + 2)(x^2 + x - 4) = 0$, then discard the solutions $s = 3$ and $s = -2$, which arise from the invalid values $a = b = c = 3$ and $a = b = c = -2$. (In the invalid cases, $s \neq a + b + c$ because $a = b = c$ is only a single root to the polynomial.)