HMMT February 2025

February 15, 2025

Combinatorics Round

1. Compute the number of ways to arrange the numbers 1, 2, 3, 4, 5, 6, and 7 around a circle such that the product of every pair of adjacent numbers on the circle is at most 20. (Rotations and reflections count as different arrangements.)

2. Kelvin the frog is on the bottom-left lily pad of a 3×3 grid of lily pads, and his home is at the top-right lily pad. He can only jump between two lily pads which are horizontally or vertically adjacent. Compute the number of ways to remove 4 of the lily pads so that the bottom-left and top-right lily pads both remain, but Kelvin cannot get home.

3. Ben has 16 balls labeled $1, 2, 3, \ldots, 16$, as well as 4 indistinguishable boxes. Two balls are *neighbors* if their labels differ by 1. Compute the number of ways for him to put 4 balls in each box such that each ball is in the same box as at least one of its neighbors. (The order in which the balls are placed does not matter.)

4. Sophie is at $(0, 0)$ on a coordinate grid and would like to get to $(3, 3)$. If Sophie is at (x, y), in a single step she can move to one of $(x+1, y)$, $(x, y+1)$, $(x-1, y+1)$, or $(x+1, y-1)$. She cannot revisit any points along her path, and neither her x-coordinate nor her y-coordinate can ever be less than 0 or greater than 3. Compute the number of ways for Sophie to reach $(3, 3)$.

5. In an 11×11 grid of cells, each pair of edge-adjacent cells is connected by a door. Karthik wants to walk a path in this grid. He can start in any cell, but he must end in the same cell he started in, and he cannot go through any door more than once (not even in opposite directions). Compute the maximum number of doors he can go through in such a path.

6. Compute the number of ways to pick two rectangles in a 5×5 grid of squares such that the edges of the rectangles lie on the lines of the grid and the rectangles do not overlap at their interiors, edges, or vertices. The order in which the rectangles are chosen does not matter.

7. Compute the number of ways to arrange 3 copies of each of the 26 lowercase letters of the English alphabet such that for any two distinct letters x_1 and x_2, the number of x_2's between the first and second occurrences of x_1 equals the number of x_2's between the second and third occurrences of x_1.

8. Albert writes 2025 numbers $a_1, \ldots, a_{2025}$ in a circle on a blackboard. Initially, each of the numbers is uniformly and independently sampled at random from the interval $[0, 1]$. Then, each second, he *simultaneously* replaces a_i with $\max(a_{i-1}, a_i, a_{i+1})$ for all $i = 1, 2, \ldots, 2025$ (where $a_0 = a_{2025}$ and $a_{2026} = a_1$). Compute the expected value of the number of distinct values remaining after 100 seconds.

9. Two points are selected independently and uniformly at random inside a regular hexagon. Compute the probability that a line passing through both of the points intersects a pair of opposite edges of the hexagon.

10. The circumference of a circle is divided into 45 arcs, each of length 1. Initially, there are 15 snakes, each of length 1, occupying every third arc. Every second, each snake independently moves either one arc left or one arc right, each with probability $\frac{1}{2}$. If two snakes ever touch, they merge to form a single snake occupying the arcs of both of the previous snakes, and the merged snake moves as one snake. Compute the expected number of seconds until there is only one snake left.