

# HMMT February 2025

February 15, 2025

## Combinatorics Round

1. Compute the number of ways to arrange the numbers 1, 2, 3, 4, 5, 6, and 7 around a circle such that the product of every pair of adjacent numbers on the circle is at most 20. (Rotations and reflections count as different arrangements.)

*Proposed by: Karthik Venkata Vedula*

**Answer:** 56

**Solution:** Fix the position of the number 7. Note that the only numbers that can be next to 7 are 1 and 2, so they must occupy the two slots adjacent to 7.

Now, the number 6 can only be adjacent to 1, 2, and 3. As 6 can no longer be adjacent to both 1 and 2, we conclude that 6 must be adjacent to 3 as well as either 1 or 2.

Finally, the numbers 4 and 5 may be placed arbitrarily in the remaining two slots.

There are 7 choices for the number 7's position, 2 ways to place 1 and 2 next to it, 2 ways to place 6 and 3, and 2 ways to place 4 and 5, giving us a total count of  $7 \cdot 2 \cdot 2 \cdot 2 = \boxed{56}$ .

2. Kelvin the frog is on the bottom-left lily pad of a  $3 \times 3$  grid of lily pads, and his home is at the top-right lily pad. He can only jump between two lily pads which are horizontally or vertically adjacent. Compute the number of ways to remove 4 of the lily pads so that the bottom-left and top-right lily pads both remain, but Kelvin cannot get home.

*Proposed by: Srinivas Arun*

**Answer:** 29

**Solution:** We instead count the arrangements for which Kelvin can get home. Note that at minimum, Kelvin must use 5 lily pads to get home, leaving 4 lily pads that are not on the path. This means that if we were to remove 4 lily pads and Kelvin can still get home, the non-removed lily pads form a shortest path from the bottom-left to the top-right. As there are  $\binom{4}{2} = 6$  of these shortest paths, our answer is  $\binom{7}{4} - 6 = \boxed{29}$ .

3. Ben has 16 balls labeled 1, 2, 3,  $\dots$ , 16, as well as 4 indistinguishable boxes. Two balls are *neighbors* if their labels differ by 1. Compute the number of ways for him to put 4 balls in each box such that each ball is in the same box as at least one of its neighbors. (The order in which the balls are placed does not matter.)

*Proposed by: Benjamin Shimabukuro*

**Answer:** 105

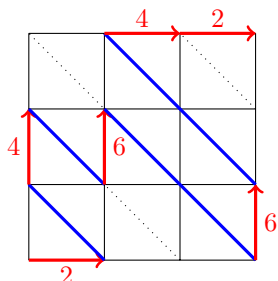
**Solution:** Each box must either contain a single group of four consecutive balls (e.g. 5, 6, 7, 8) or two groups of two consecutive balls (e.g. 5, 6, 9, 10). Since all groups have even lengths, this means that 1 and 2 are in the same group, 3 and 4 are in the same group, and so on. We can think of each of these 8 pairs of balls as an individual unit, so the answer is equal to the number of ways to put 8 objects in 4 indistinguishable boxes, where each box has 2 objects without any additional restrictions. The number of ways to do this is  $\frac{8!}{2^4 \cdot 4!} = \boxed{105}$ .

4. Sophie is at  $(0, 0)$  on a coordinate grid and would like to get to  $(3, 3)$ . If Sophie is at  $(x, y)$ , in a single step she can move to one of  $(x + 1, y)$ ,  $(x, y + 1)$ ,  $(x - 1, y + 1)$ , or  $(x + 1, y - 1)$ . She cannot revisit any points along her path, and neither her  $x$ -coordinate nor her  $y$ -coordinate can ever be less than 0 or greater than 3. Compute the number of ways for Sophie to reach  $(3, 3)$ .

Proposed by: Derek Liu

Answer:

**Solution:** Let a *lateral* move refer to one which is either up or right. Then the lateral moves are the only ones which increase Kelvin's sum of coordinates by 1, while all other moves do not change the sum, so Kelvin must make 6 of them, one to increase this sum from  $i$  to  $i + 1$  for each  $i \in [0, 5]$ .



We claim there exists a unique path corresponding to each set of 6 lateral moves chosen in this way. Indeed, the diagonal moves allow Kelvin to get from any point on  $x + y = i$  to any second point on  $x + y = i$  in exactly one way.

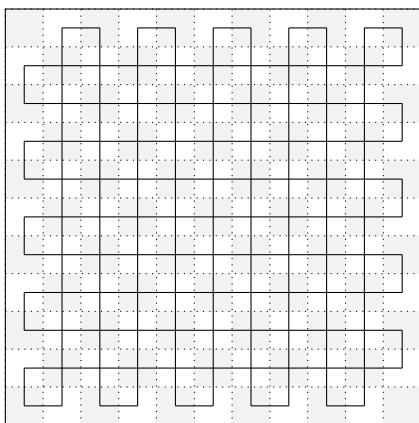
Observe that when  $i \leq 2$ , the number of lateral moves that increase the sum of coordinates from  $i$  to  $i + 1$  is  $2i$ , as is the number that increase it from  $5 - i$  to  $6 - i$ . Thus the answer is  $2 \cdot 4 \cdot 6 \cdot 6 \cdot 4 \cdot 2 = \boxed{2304}$ .

5. In an  $11 \times 11$  grid of cells, each pair of edge-adjacent cells is connected by a door. Karthik wants to walk a path in this grid. He can start in any cell, but he must end in the same cell he started in, and he cannot go through any door more than once (not even in opposite directions). Compute the maximum number of doors he can go through in such a path.

Proposed by: Derek Liu

Answer:

**Solution:**



This is simply asking for the longest circuit in the adjacency graph of this grid. Note that this grid has  $4 \cdot 9 = 36$  cells of odd degree, 9 along each side. If we color the cells with checkerboard colors so that the corners are black, then 20 of these 36 cells are white. An Eulerian circuit uses an even number of doors from each cell, so at least one door from each of these cells goes unused. No door connects two white cells, so at least 20 doors are unused, leaving at most  $2 \cdot 10 \cdot 11 - 20 = \boxed{200}$  doors crossed.

To see this is achievable, we first delete the bottom-right cell and its 2 doors, as well as the top-left cell and its 2 doors. This leaves 8 odd-degree cells along each side of the grid; we can delete 4 doors along each to cover all remaining odd-degree cells, for 20 total doors deleted. The resulting graph is connected and has no odd-degree cells, so it must have an Eulerian circuit. This circuit is our desired path.

6. Compute the number of ways to pick two rectangles in a  $5 \times 5$  grid of squares such that the edges of the rectangles lie on the lines of the grid and the rectangles do not overlap at their interiors, edges, or vertices. The order in which the rectangles are chosen does not matter.

*Proposed by: Jacob Paltrowitz*

**Answer:** 6300

**Solution:** A rectangle can be specified by two intervals, one specifying its horizontal extent ( $x$ -coordinates of left and right sides) and one specifying its vertical extent ( $y$ -coordinates of bottom and top sides). For the rectangles to not overlap, we need either the horizontal intervals or the vertical intervals to be disjoint (possibly both).

First, we will count the number of ways for the horizontal intervals to be disjoint. Let these intervals be  $[a, b]$  and  $[c, d]$ . As the order of the rectangles does not matter, we can assume without loss of generality that  $a < c$ , so  $a < b < c < d$ . Then there are  $\binom{6}{4}$  choices for  $a, b, c$ , and  $d$ . There are no restrictions on the vertical intervals, so the number of ways to choose them is  $\binom{6}{2}^2$ . Thus, the total number of pairs of rectangles for which the horizontal intervals are disjoint is  $\binom{6}{4} \binom{6}{2}^2$ .

By symmetry, the total number of pairs of rectangles for which the vertical intervals are disjoint is the same. It remains to count the number of ways for both the horizontal and vertical intervals to be disjoint. Again, let the horizontal intervals be  $[a, b]$  and  $[c, d]$ , and let the vertical intervals be  $[e, f]$  and  $[g, h]$ . We can assume  $a < b < c < d$  without loss of generality, so there are  $\binom{6}{4}$  ways to choose the horizontal intervals. However, the cases  $e < f < g < h$  and  $g < h < e < f$  are now distinct, so there are  $2\binom{6}{4}$  ways to choose the vertical intervals. Therefore, there are  $2\binom{6}{4}^2$  pairs of rectangles for which both the horizontal and vertical intervals are disjoint.

Through inclusion-exclusion, we get the final answer of

$$2 \cdot \binom{6}{4} \binom{6}{2}^2 - 2 \cdot \binom{6}{4}^2 = \boxed{6300}.$$

7. Compute the number of ways to arrange 3 copies of each of the 26 lowercase letters of the English alphabet such that for any two distinct letters  $x_1$  and  $x_2$ , the number of  $x_2$ 's between the first and second occurrences of  $x_1$  equals the number of  $x_2$ 's between the second and third occurrences of  $x_1$ .

*Proposed by: Derek Liu*

**Answer:**  $2^{25}26!$

**Solution:** First, we prove such a string can be divided into blocks where each block consists of the same substring written three times. We prove the following lemma.

**Lemma 1.** For any letter  $x_1$ , the strings between the first and second occurrences of  $x_1$  and between the second and third occurrences of  $x_1$  are the same.

*Proof.* Call these two strings  $s_1$  and  $s_2$ . We know they must be permutations of each other, and if a letter appears twice in  $s_1$ , it would also have to appear twice in  $s_2$ , for four appearances in all, which is impossible. Thus, no letter appears twice in  $s_1$  (and likewise in  $s_2$ ).

Assume for sake of contradiction that for some letters  $x_2$  and  $x_3$  in these strings,  $x_2$  appears before  $x_3$  in  $s_1$ , but after  $x_3$  in  $s_2$ . Then, between these two appearances of  $x_2$  (which are consecutive, because

no other  $x_2$ 's appear in either  $s_1$  or  $s_2$ ), there must be two  $x_3$ 's. This implies there must also be two  $x_3$ 's between the other pair of consecutive  $x_2$ 's, contradiction.

We conclude any two letters in  $s_1$  and  $s_2$  appear in the same order in both strings, so  $s_1 = s_2$ .  $\square$

Let the first letter of our 78-character string be  $x_1$ , and suppose the next appearance of  $x_1$  is the  $(k+1)$ -th letter. Let  $x_2, x_3, \dots, x_k$  be the letters in between. Then,  $x_2x_3 \dots x_k$  is the string between the first and second  $x_1$ 's, so it must also be the string between the second and third  $x_1$ 's. Thus, after the second  $x_1$ , we must have  $x_2x_3 \dots x_kx_1$ .

Now, between the first and second  $x_k$ 's is the string  $x_1x_2x_3 \dots x_{k-1}$ , so this must also be between the second and third  $x_k$ 's. Thus, after the second  $x_k$ , we must have  $x_1x_2x_3 \dots x_k$ .

Thus, the first  $3k$  letters are simply  $x_1x_2x_3 \dots x_k$  repeated three times. We can remove this block of  $3k$  letters and repeat to show that the whole string can be divided into such blocks.

To count the number of such strings, we first note that there are  $2^{25}$  ways to divide the strings into such blocks. This is because there are 25 possible places which can divide two blocks (after the 3rd, 6th, 9th, etc. letters), and we can choose any subset of these to divide blocks.

The string is then uniquely determined by the first one-third of each block, which must consist of every letter exactly once (as the whole string is just three copies of these thirds spliced together). These thirds can consist of any ordering of the 26 letters, so there are  $26!$  strings with any given partition of blocks.

We conclude the total number of strings is  $\boxed{2^{25} \cdot 26!}$ .

*Remark.* Another way to see the characterization of strings is as follows. First, solve the problem for the case of two letters  $a$  and  $b$ : it is straightforward to show that the only strings are  $ababab$ ,  $bababa$ ,  $aaabbb$ ,  $bbbaaa$ . Going back to the original problem, the six appearances of any pair of letters must be in the order of one of the four strings above; say two letters are *interleaved* if they alternate appearances. Construct a graph on letters with an edge between two letters if and only if they are interleaved. Note that if  $a$  and  $b$  are interleaved, and  $b$  and  $c$  are interleaved, then  $a$  and  $c$  must be interleaved. Thus, the graph is transitive, implying that it is a union of disjoint cliques. We can now finish as before.

8. Albert writes 2025 numbers  $a_1, \dots, a_{2025}$  in a circle on a blackboard. Initially, each of the numbers is uniformly and independently sampled at random from the interval  $[0, 1]$ . Then, each second, he *simultaneously* replaces  $a_i$  with  $\max(a_{i-1}, a_i, a_{i+1})$  for all  $i = 1, 2, \dots, 2025$  (where  $a_0 = a_{2025}$  and  $a_{2026} = a_1$ ). Compute the expected value of the number of distinct values remaining after 100 seconds.

*Proposed by: Isaac Zhu*

**Answer:**  $\boxed{\frac{2025}{101}}$

**Solution:** We can assume that the initial numbers are all distinct, since this occurs with probability 1. For clarity, we denote the value of  $a_i$  after  $t$  seconds as  $a_{i,t}$ . The index  $i$  is taken mod 2025.

In general, after  $k < 1012$  seconds, we claim the expected number of distinct values remaining is  $\frac{2025}{k+1}$ . To show this, we first prove that for any remaining value, its appearances are consecutive. Indeed, note that for all  $i$  and  $k$ ,

$$a_{i,k} = \max(a_{i-1,k-1}, a_{i,k-1}, a_{i+1,k-1}) = \max(a_{i-2,k-2}, \dots, a_{i+2,k-2}) = \dots = \max(a_{i-k,0}, \dots, a_{i+k,0}).$$

Given an initial number  $a_{c,0}$ , let  $j_1$  and  $j_2$  be the smallest positive integers such that  $a_{c-j_1,0} > a_{c,0}$  and  $a_{c+j_2,0} > a_{c,0}$ . As the initial numbers are distinct, we conclude  $a_{i,k} = a_{c,0}$  if and only if  $\{i-k, i-k+1, \dots, i+k\}$  contains  $c$  but neither  $c-j_1$  nor  $c+j_2 \pmod{2025}$ . The indices  $i$  that satisfy this are clearly consecutive.

Now consider the indicator variables

$$\mathbf{1}_i = \begin{cases} 1 & \text{if } a_{i,k} \neq a_{i+1,k} \\ 0 & \text{otherwise} \end{cases}$$

for  $1 \leq i \leq 2025$ . Note that the number of distinct values on the board after  $k$  seconds is simply  $\sum_{i=1}^{2025} \mathbf{1}_i$ . By linearity of expectation, it suffices to compute  $\mathbb{E}[\mathbf{1}_i]$  for each  $i$ . Recall

$$a_{i,k} = \max(a_{i-k,0}, \dots, a_{i+k,0})$$

for each  $1 \leq i \leq 2025$ . Hence, the condition  $a_{i,k} \neq a_{i+1,k}$  can be written as

$$\max(a_{i-k,0}, \dots, a_{i+k,0}) \neq \max(a_{i+1-k,0}, \dots, a_{i+1+k,0}).$$

This is the case if and only if  $\max(a_{i-k,0}, \dots, a_{i+k+1,0})$  is either  $a_{i-k,0}$  or  $a_{i+k+1,0}$  (as  $a_{i-k,0} \neq a_{i+k+1,0}$  and  $2k+2 < 2025$ ). Since the  $2k+2$  values  $a_{i-k,0}, \dots, a_{i+k+1,0}$  were sampled independently from the same distribution, each of  $a_{i-k,0}$  and  $a_{i+k+1,0}$  has a  $\frac{1}{2k+2}$  probability of being their maximum. Hence,

$$\mathbb{E}[\mathbf{1}_i] = \Pr[a_{i,k} \neq a_{i+1,k}] = \frac{2}{2k+2} = \frac{1}{k+1}.$$

Thus, the expected number of distinct values is

$$\sum_{i=1}^{2025} \mathbb{E}[\mathbf{1}_i] = \frac{2025}{k+1}.$$

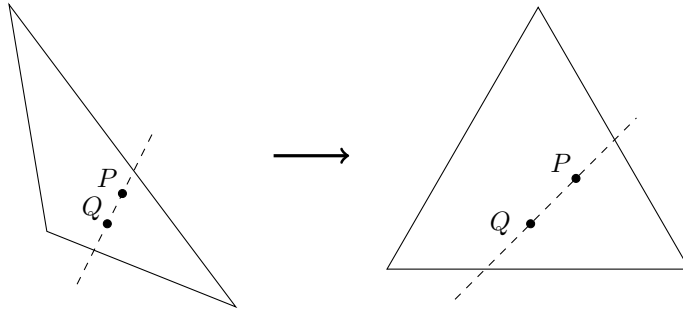
Substituting  $k = 100$  yields the answer  $\boxed{\frac{2025}{101}}$ .

9. Two points are selected independently and uniformly at random inside a regular hexagon. Compute the probability that a line passing through both of the points intersects a pair of opposite edges of the hexagon.

*Proposed by: Albert Wang*

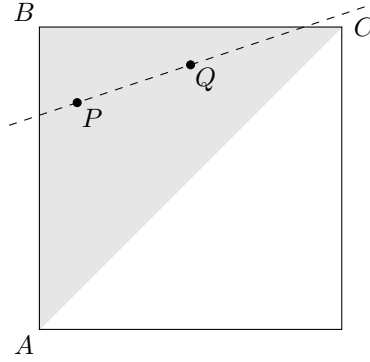
**Answer:**  $\boxed{\frac{4}{9}}$

**Solution:**

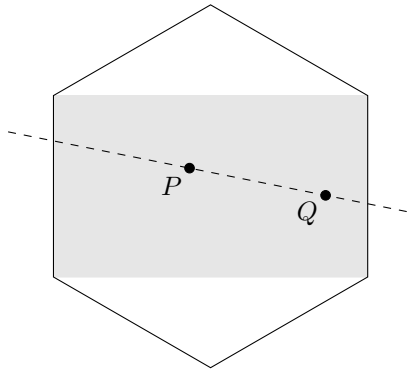


First, we compute the probability that the line through two random points in a triangle  $ABC$  passes through segments  $\overline{AB}$  and  $\overline{AC}$ . We can take an affine transform of the two random points and the triangle such that  $ABC$  becomes equilateral. Since the distribution of the two points is still uniform and independent, the probability of the line intersecting any two given sides is  $\frac{1}{3}$  by symmetry.

Next, we compute the probability that the line through two random points in a rectangle  $ABCD$  passes through opposite edges  $\overline{AB}$  and  $\overline{CD}$ .



If the line passes through  $\overline{AB}$  and  $\overline{BC}$ , the points must both lie in triangle  $ABC$ , whose area is half that of  $ABCD$ . Given this, the probability the line passes through those two sides is  $\frac{1}{3}$ , as computed before. Thus the probability the line passes through  $\overline{AB}$  and  $\overline{BC}$  is  $\left(\frac{1}{2}\right)^2 \cdot \frac{1}{3} = \frac{1}{12}$ . The same goes for the other pairs of adjacent edges. By symmetry, the line is equally likely to pass through either pair of opposite edges, each with probability  $\frac{1}{2} \left(1 - 4 \cdot \frac{1}{12}\right) = \frac{1}{3}$ .



We now return to the original problem. If the line passes through a pair of opposite edges, then both points must be in the rectangle formed by these edges, which has area  $\frac{2}{3}$  that of the hexagon. Given this, the probability the line passes through those two edges is  $\frac{1}{3}$  as computed before. Thus, the probability that the line passes through the given pair of opposite edges is  $\left(\frac{2}{3}\right)^2 \cdot \frac{1}{3} = \frac{4}{27}$ . Hence, the probability the line passes through any of the three pairs of opposite edges is  $3 \cdot \frac{4}{27} = \boxed{\frac{4}{9}}$ .

10. The circumference of a circle is divided into 45 arcs, each of length 1. Initially, there are 15 snakes, each of length 1, occupying every third arc. Every second, each snake independently moves either one arc left or one arc right, each with probability  $\frac{1}{2}$ . If two snakes ever touch, they merge to form a single snake occupying the arcs of both of the previous snakes, and the merged snake moves as one snake. Compute the expected number of seconds until there is only one snake left.

*Proposed by: Srinivas Arun*

**Answer:**  $\boxed{\frac{448}{3}}$

**Solution:** We solve the problem generally for  $n$  snakes and  $3n$  arcs. Without loss of generality, fix the two snakes  $A$  and  $B$  that will eventually form the ends of the last snake. Note that  $A$  and  $B$  must be consecutive in the initial configuration; assume  $B$  lies immediately clockwise from  $A$ .

Let  $d$  be the arclength between  $A$  and  $B$  (measuring clockwise from  $A$  to  $B$ ). As long as  $d \neq 0$  and  $d \neq 2n$ , we know that  $A$  and  $B$  lie in different snakes and thus move independently. Therefore, we can consider  $d$  to be on a random walk starting at 2, where

- the probability  $d$  does not change is  $\frac{1}{2}$  (if  $A$  and  $B$  move in the same direction),
- the probability  $d$  decreases by 2 is  $\frac{1}{4}$  (if  $A$  moves clockwise and  $B$  moves counterclockwise),
- the probability that  $d$  increases by 2 is  $\frac{1}{4}$  (if  $A$  moves counterclockwise and  $B$  moves clockwise),
- the random walk ends if  $d = 0$  or  $d = 2n$  (the snakes touch; only  $d = 2n$  obeys our assumption).

We want to find the conditional expected length of the random walk given that it ends at  $d = 2n$ .

The key idea is to directly calculate the conditional probabilities of each transition. To do this, we will repeatedly use the following well-known fact: for any random walk where moving 1 unit to the left and moving 1 unit to the right are equally likely, the probability of reaching the point  $b$  units to the right of the starting point before reaching the point  $a$  units to the left is  $\frac{a}{a+b}$ .

For any single transition, we have

$$\begin{aligned}
& \mathbb{P}(\text{"}d \text{ moves from } 2k \text{ to } 2k-2 \text{"} \mid \text{"the path ends at } 2n\text{"}) \\
&= \frac{\mathbb{P}(\text{"}d \text{ moves from } 2k \text{ to } 2k-2 \text{" AND "the path starting from } 2k-2 \text{ ends at } 2n\text{"})}{\mathbb{P}(\text{"the path starting from } 2k \text{ ends at } 2n\text{"})} \\
&= \frac{\left(\frac{1}{4}\right)\binom{k-1}{n}}{\binom{k}{n}} = \frac{k-1}{4k},
\end{aligned}$$

$$\begin{aligned}
& \mathbb{P}(\text{"}d \text{ moves from } 2k \text{ to } 2k+2 \text{"} \mid \text{"the path ends at } 2n\text{"}) \\
&= \frac{\mathbb{P}(\text{"}d \text{ moves from } 2k \text{ to } 2k+2 \text{" AND "the path starting from } 2k+2 \text{ ends at } 2n\text{"})}{\mathbb{P}(\text{"the path starting from } 2k \text{ ends at } 2n\text{"})} \\
&= \frac{\left(\frac{1}{4}\right)\binom{k+1}{n}}{\binom{k}{n}} = \frac{k+1}{4k},
\end{aligned}$$

$$\begin{aligned}
& \mathbb{P}(\text{"}d \text{ stays at } 2k \text{"} \mid \text{"the path ends at } 2n\text{"}) \\
&= \frac{\mathbb{P}(\text{"}d \text{ stays at } 2k \text{" AND "the path starting from } 2k \text{ ends at } 2n\text{"})}{\mathbb{P}(\text{"the path starting from } 2k \text{ ends at } 2n\text{"})} \\
&= \frac{\left(\frac{1}{2}\right)\binom{k}{n}}{\binom{k}{n}} = \frac{1}{2}.
\end{aligned}$$

Let  $E_{2k}$  denote the expected length of a walk starting at  $2k$  given that the walk ends at  $2n$ . Then, for  $0 < k < n$ , we have

$$\begin{aligned}
E_{2k} &= \frac{k-1}{4k}(E_{2k-2} + 1) + \frac{1}{2}(E_{2k} + 1) + \frac{k+1}{4k}(E_{2k+2} + 1) \\
\implies E_{2k} &= 2 + \frac{k-1}{2k}E_{2k-2} + \frac{k+1}{2k}E_{2k+2}.
\end{aligned}$$

It can be shown by induction that  $E_{2k} = E_2 - \frac{2(k^2-1)}{3}$  for all  $k \geq 1$ . Since  $E_{2n} = 0$ , this implies that  $E_2 = \frac{2(n^2-1)}{3}$ . When  $n = 15$ , the answer is therefore  $\boxed{\frac{448}{3}}$ .