HMMT February 2025

February 15, 2025

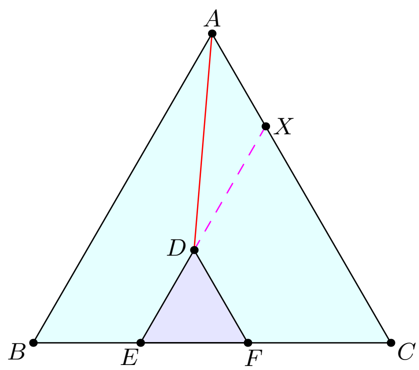
Geometry Round

1. Equilateral triangles $\triangle ABC$ and $\triangle DEF$ are drawn such that points B, E, F, and C lie on a line in this order, and point D lies inside triangle $\triangle ABC$. If $BE = 14$, $EF = 15$, and $FC = 16$, compute AD.

   Proposed by: Jackson Dryg, Karthik Venkata Vedula

   Answer: $\boxed{26}$

   Solution:



   Extend DE to meet AC at X. Observe that $ABEX$ and $DFCX$ are isosceles trapezoids (both with base angles of $60°$), so we have

   - $AX = BE = 14$,
   - $DX = FC = 16$, and
   - $\angle AXD = 120°$.

   By Law of Cosines on $\triangle ADX$, the answer is

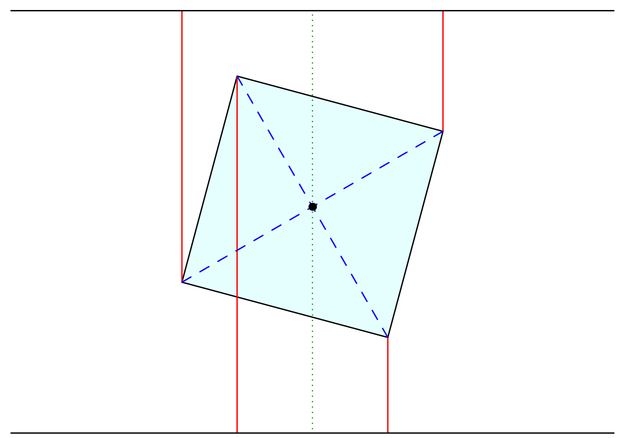
$$\begin{aligned} AD &= \sqrt{AX^2 + DX^2 - 2\cos(120°)AX \cdot DX} \\ &= \sqrt{14^2 + 16^2 + 14 \cdot 16} = \boxed{26}. \end{aligned}$$

2. In a two-dimensional cave with a parallel floor and ceiling, two stalactites of lengths 16 and 36 hang perpendicularly from the ceiling, while two stalagmites of heights 25 and 49 grow perpendicularly from the ground. If the tips of these four structures form the vertices of a square in some order, compute the height of the cave.

   Proposed by: Albert Wang

   Answer: $\boxed{63}$

   Solution: Note that the difference in heights between the two stalactites does not equal the difference in heights between the stalagmites. This tells us that the two stalactites form a pair of opposite vertices of the square, and likewise for the stalagmites. As the midpoint of the pairs of structures must then coincide, we know that it is $\frac{16+36}{2} = 26$ from the ceiling due to the stalactites, and $\frac{25+49}{2} = 37$ from the ground due to the stalagmites. Therefore the height of the cave is just the sum of these two values, i.e. $\boxed{63}$.

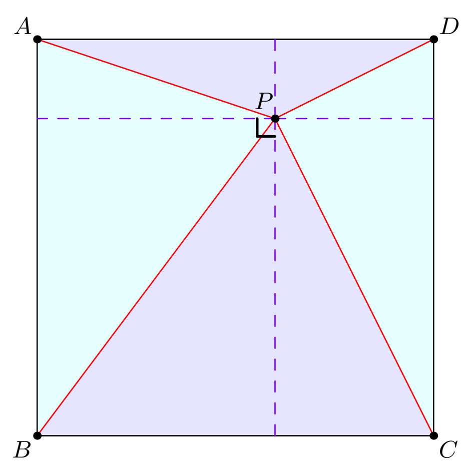


3. Point P lies inside square $ABCD$ such that the areas of $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, and $\triangle PDA$ are 1, 2, 3, and 4, in some order. Compute $PA \cdot PB \cdot PC \cdot PD$.

   Proposed by: Karthik Venkata Vedula

   Answer: $\boxed{8\sqrt{10}}$

   Solution:



Let h_1, h_2, h_3, and h_4 be the lengths of the altitudes from P to sides AB, BC, CD, and DA, respectively. Then, the problem statement implies that $\{h_1, h_2, h_3, h_4\} = \{x, 2x, 3x, 4x\}$ for some x. Furthermore, the area of the square is $1 + 2 + 3 + 4 = 10$, so we have

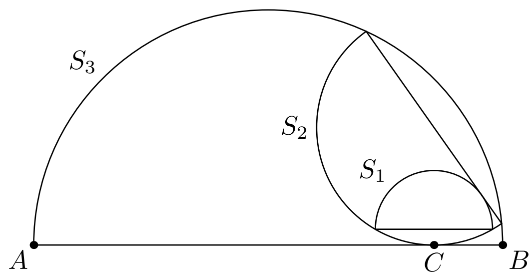
$$h_1 + h_3 = \sqrt{10} = h_2 + h_4.$$

Hence either ($\{h_1, h_3\} = \{x, 4x\}$ and $\{h_2, h_4\} = \{2x, 3x\}$), or ($\{h_2, h_4\} = \{x, 4x\}$ and $\{h_1, h_3\} = \{2x, 3x\}$). In any case, we get $5x = \sqrt{10}$ from above, and finish via the Pythagorean theorem:

$$\begin{aligned} PA \cdot PB \cdot PC \cdot PD &= \sqrt{x^2 + (2x)^2} \cdot \sqrt{(2x)^2 + (4x)^2} \cdot \sqrt{(4x)^2 + (3x)^2} \cdot \sqrt{(3x)^2 + x^2} \\ &= 50\sqrt{10}x^4 \\ &= (50\sqrt{10}) \cdot \left(\frac{2}{5}\right)^2 \\ &= \boxed{8\sqrt{10}}. \end{aligned}$$

4. A semicircle is inscribed in another semicircle if the smaller semicircle's diameter is a chord of the larger semicircle, and the smaller semicircle's arc is tangent to the diameter of the larger semicircle.

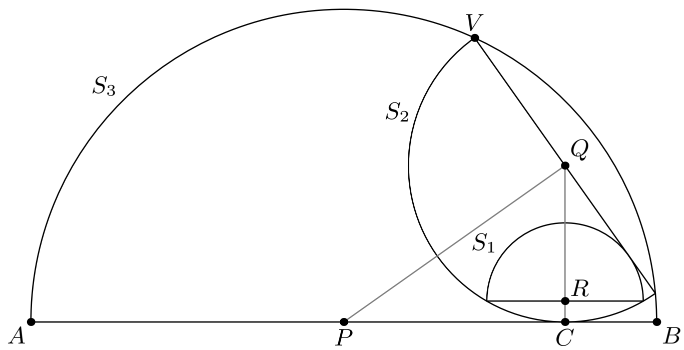
   Semicircle S_1 is inscribed in a semicircle S_2, which is inscribed in another semicircle S_3. The radii of S_1 and S_3 are 1 and 10, respectively, and the diameters of S_1 and S_3 are parallel. The endpoints of the diameter of S_3 are A and B, and S_2's arc is tangent to AB at C. Compute $AC \cdot CB$.



   Proposed by: Karthik Venkata Vedula

   Answer: $\boxed{20}$

   Solution:



   Let P, Q, and R be the midpoints of the diameters (i.e., the center of the circular arcs) of S_3, S_2, and S_1, respectively. Observe that if one fixes S_3, the location of S_2 is uniquely determined by the angle between the diameters of S_2 and S_3. The same holds for S_2 and S_1. Thus, the figures $S_3 \cup S_2$ and $S_2 \cup S_1$ are similar. This gives us that the radius of S_2 is $\sqrt{10}$.

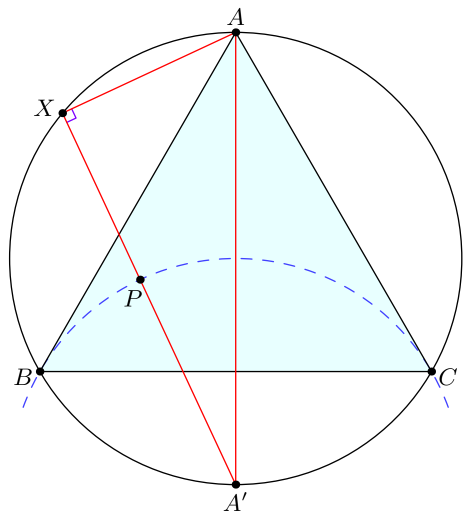
   To compute the answer, we define V to be either intersection of the arcs of S_2 and S_3. By the Pythagorean theorem, $PQ = \sqrt{PV^2 - VQ^2} = \sqrt{100 - 10} = \sqrt{90}$. By the Pythagorean theorem again, $PC = \sqrt{PQ^2 - QC^2} = \sqrt{90 - 10} = \sqrt{80}$. Thus $AC \cdot CB = (10 + PC)(10 - PC) = 100 - PC^2 = \boxed{20}$.

5. Let $\triangle ABC$ be an equilateral triangle with side length 6. Let P be a point inside triangle $\triangle ABC$ such that $\angle BPC = 120°$. The circle with diameter $\overline{AP}$ meets the circumcircle of $\triangle ABC$ again at $X \neq A$. Given that $AX = 5$, compute XP.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{\sqrt{23} - 2\sqrt{3}}$

   Solution:



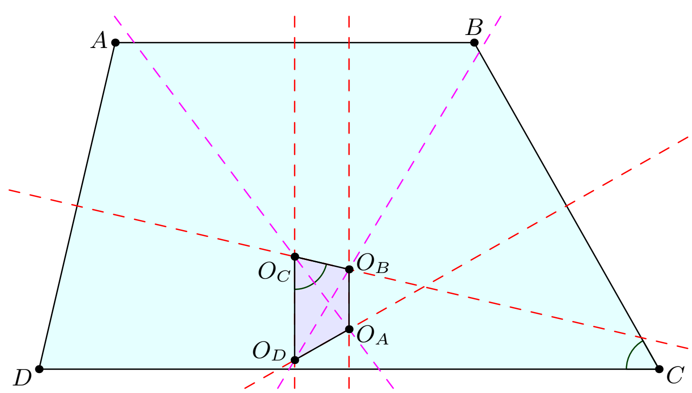
Let A' be the antipode of A. As $\angle AXA' = 90^\circ$, we have X, P, and A' are collinear. As $\angle BPC = 120^\circ$ and $\angle BA'C = 180^\circ - \angle BAC = 120^\circ$, it follows that P lies on the circle with center A' passing through B and C, so $A'P = \frac{BC}{\sqrt{3}} = 2\sqrt{3}$ and $AA' = 2A'B = 4\sqrt{3}$. By the Pythagorean theorem, $XA' = \sqrt{(4\sqrt{3})^2 - 5^2} = \sqrt{23}$, so the answer is $XA' - PA' = \boxed{\sqrt{23} - 2\sqrt{3}}$.

6. Trapezoid $ABCD$, with $AB \parallel CD$, has side lengths $AB = 11$, $BC = 8$, $CD = 19$, and $DA = 4$. Compute the area of the convex quadrilateral whose vertices are the circumcenters of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$.

   Proposed by: Karthik Venkata Vedula, Pitchayut Saengrungkongka

   Answer: $\boxed{9\sqrt{15}}$

   Solution:



Let O_A, O_B, O_C, and O_D be the circumcenters of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Note that O_BO_C is the perpendicular bisector of $\overline{AD}$. Similarly, $O_BO_D \perp AC$ and $O_CO_D \perp AB$. As $AB \parallel CD$, we have $O_CO_D \perp CD$. Then, $\triangle O_BO_CO_D \overset{+}{\sim} \triangle ADC$, as their corresponding sides are perpendicular. Likewise, $\triangle O_DO_AO_B \overset{+}{\sim} \triangle CBA$, so $O_AO_BO_CO_D \overset{+}{\sim} BADC$.

Therefore, we only need to compute the area of $ABCD$ and the ratio of similarity between the two trapezoids. Draw a line parallel to $\overline{AD}$ passing through B. Let this line intersect $\overline{CD}$ at X. Then, $BX = AD = 4$, $CB = 8$, and $CX = CD - AB = 8$. The height h of $ABCD$ is given by

$$h = d(B, \overline{XC}) = \frac{2[\triangle BXC]}{XC} = \frac{BX \cdot d(C, \overline{BX})}{XC} = \frac{4\sqrt{8^2 - 2^2}}{8} = \sqrt{15}.$$

On the other hand, the height of $O_AO_BO_CO_D$ is the distance between the perpendicular bisectors of $\overline{AB}$ and $\overline{CD}$, which pass through the midpoints M of $\overline{AB}$ and N of $\overline{CD}$. Let A' and M' be the projections of A and M onto $\overline{CD}$, respectively. Then, the height of $O_AO_BO_CO_D$ is given by

$$M'N = DN - DM' = \frac{CD}{2} - \frac{AB}{2} - DA' = \frac{19-11}{2} - \sqrt{4^2 - h^2} = 3.$$

Therefore, the similarity ratio between the two trapezoids is $\frac{3}{\sqrt{15}}$. We know that the area of $ABCD$ is $\frac{1}{2}(11+19)\sqrt{15} = 15\sqrt{15}$, so the area of $O_AO_BO_CO_D$ is $\left(\frac{3}{\sqrt{15}}\right)^2 \cdot 15\sqrt{15} = \boxed{9\sqrt{15}}$.

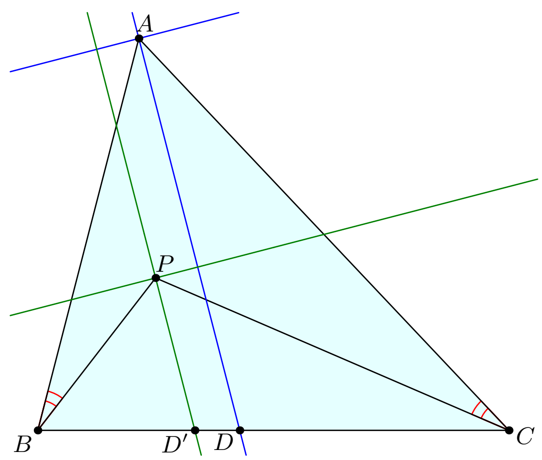
7. Point P is inside triangle $\triangle ABC$ such that $\angle ABP = \angle ACP$. Given that $AB = 6$, $AC = 8$, $BC = 7$, and $\frac{BP}{PC} = \frac{1}{2}$, compute $\frac{[BPC]}{[ABC]}$.

   (Here, $[XYZ]$ denotes the area of $\triangle XYZ$).

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{\frac{7}{18}}$

   Solution 1:



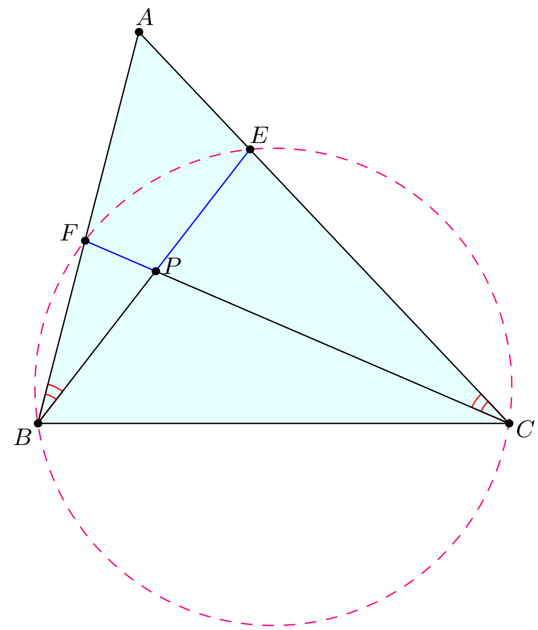
Let the internal and external bisectors of $\angle BAC$ meet BC at D and E. Similarly, let the internal and external bisectors of $\angle BPC$ meet BC at D' and E'. The angle condition implies that $AD \parallel PD'$ and $AE \parallel PE'$. Thus, triangles ADE and $PD'E'$ are homothetic. Hence, the requested ratio is $\frac{DE}{D'E'}$.

Repeated applications of the angle bisector theorem yield

$$\begin{aligned} BD &= 7 \cdot \tfrac{6}{6+8} = 3, \\ BE &= 7 \cdot \tfrac{6}{8-6} = 21, \\ BD' &= 7 \cdot \tfrac{1}{1+2} = \tfrac{7}{3}, \\ BE' &= 7 \cdot \tfrac{1}{2-1} = 7, \end{aligned}$$

so $DE = 24$ and $D'E' = 28/3$. Hence, the answer is $\frac{28/3}{24} = \boxed{\frac{7}{18}}$.

Solution 2:



Let $D = AP \cap BC$, $E = BP \cap AC$ and $F = CP \cap AB$. Then, $BCEF$ is cyclic, so by Power of a Point, $\frac{AE}{AF} = \frac{AB}{AC} = \frac{3}{4}$. Let $AE = 3x$ and $AF = 4x$. Then,

$$\triangle PFB \sim \triangle PEC \implies \frac{BF}{CE} = \frac{BP}{CP} \implies \frac{6-4x}{8-3x} = \frac{1}{2}.$$

Solving for x gives $x = \frac{4}{5}$, so we get that $AF = \frac{16}{5}$, $FB = \frac{14}{5}$, $AE = \frac{12}{5}$, and $EC = \frac{40}{5}$. By Ceva's theorem on $\triangle ABC$ and point P, we have

$$\frac{BD}{DC} = \frac{AE}{EC} \cdot \frac{BF}{FA} = \frac{AE}{AF} \cdot \frac{BF}{CE} = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}.$$

Finally, by Menelaus's theorem on $\triangle ADC$ and line BE, we get that

$$\frac{AP}{PD} = \frac{CB}{BD} \cdot \frac{AE}{EC} = \frac{11}{3} \cdot \frac{12}{28} = \frac{11}{7},$$

which implies that $\frac{[BPC]}{[ABC]} = \boxed{\frac{7}{18}}$.

Solution 3: We use barycentric coordinates with respect to $\triangle ABC$. From $\angle ABP = \angle ACP$, we get

$$\frac{[ABP]}{[ACP]} = \frac{AB \cdot BP}{AC \cdot CP} = \frac{3}{8},$$

so P has coordinate $(- : 8 : 3)$. Let Q be the isogonal conjugate of P, so Q lies on perpendicular bisector of BC. By Steiner ratio theorem, Q has coordinate $\left(- : 8^2/8 : 6^2/3\right) = (- : 8 : 12) = (- : 2 : 3)$. Let the coordinate be $(t : 2 : 3)$ for some real number t. Then, recall the equation for the perpendicular bisector (from Corollary 6 of `https://web.evanchen.cc/handouts/bary/bary-full.pdf`):

$$\begin{aligned} a^2(z-y) + x(c^2-b^2) &= 0 \\ \implies 7^2(3-2) + t(6^2-8^2) &= 0, \end{aligned}$$

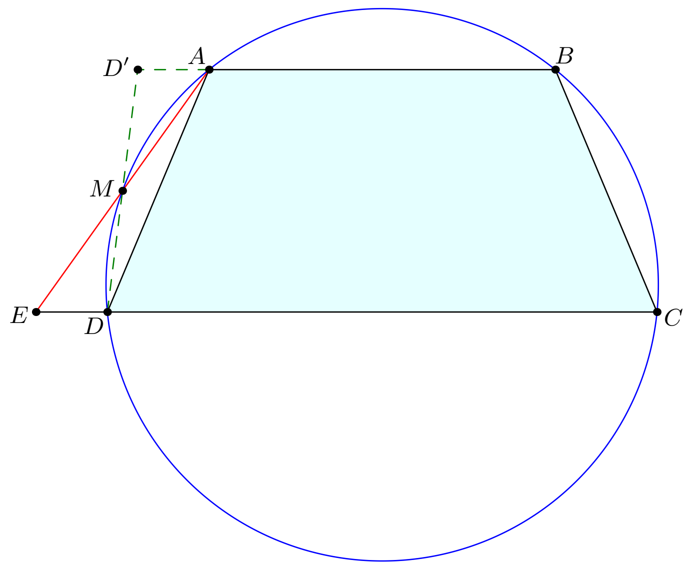
so $t = \frac{7^2(3-2)}{8^2-6^2} = \frac{7}{4}$. Hence, point Q has coordinates $(7 : 8 : 12)$, so point P has coordinates $(7^2/7 : 8^2/8 : 6^2/12) = (7 : 8 : 3)$. Therefore, $[BPC]/[ABC] = \boxed{\frac{7}{18}}$.

8. Let $ABCD$ be an isosceles trapezoid such that $CD > AB = 4$. Let E be a point on line CD such that $DE = 2$ and D lies between E and C. Let M be the midpoint of $\overline{AE}$. Given that points A, B, C, D, and M lie on a circle with radius 5, compute MD.

   Proposed by: Sarunyu Thongjarast

   Answer: $\boxed{\sqrt{6}}$

   Solution:



Let D' be the reflection of D across M. Then, $ADED'$ is a parallelogram. Hence, $D'A = 2$, so $D'B = 6$. Thus, if $D'M = MD = x$, then Power of a Point at D' gives $x \cdot (2x) = 2 \cdot 6$, so $x = \boxed{\sqrt{6}}$.

Remark. The radius of the circle is unnecessary.

9. Let $ABCD$ be a rectangle with $BC = 24$. Point X lies inside the rectangle such that $\angle AXB = 90°$. Given that triangles $\triangle AXD$ and $\triangle BXC$ are both acute and have circumradii 13 and 15, respectively, compute AB.

   Proposed by: Pitchayut Saengrungkongka

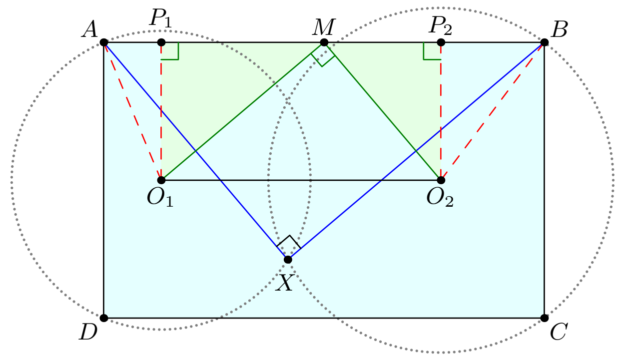
   Answer: $\boxed{14 + 4\sqrt{37}.}$

   Solution 1: Let M be the midpoint of AB. Let O_1 and O_2 be the circumcenters of $\triangle AXD$ and $\triangle BXC$, respectively. Since O_1M is the perpendicular bisector of AX and O_2M is the perpendicular bisector of BX, we get that $\angle O_1MO_2 = 90°$.

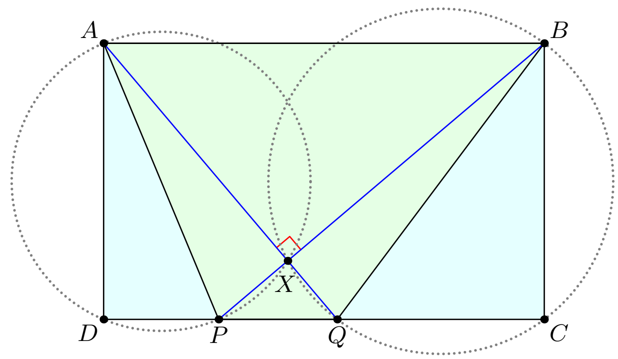
   Let P_1 and P_2 be the projections of O_1 and O_2 onto segment AB, respectively, and let $AB = 2x$. By the Pythagorean theorem, $P_1A = \sqrt{O_1A^2 - O_1P_1^2} = \sqrt{15^2 - 12^2} = 5$, so $MP_1 = MA - P_1A = x - 5$. Likewise, $MP_2 = MB - \sqrt{15^2 - 12^2} = x - 9$. Since $\triangle MP_1O_1 \sim \triangle O_2P_2M$, we know

$$(x-5)(x-9) = MP_1 \cdot MP_2 = O_2P_2 \cdot P_1O_1 = 12^2.$$

   Solving this, we get $x = 7 + 2\sqrt{37}$, which implies that $AB = 2x = \boxed{14 + 4\sqrt{37}}$. (The condition that $\triangle AXD$ and $\triangle BXC$ are acute rules out $14 - 4\sqrt{37}$.)



Solution 2:



Let P be the antipode of A in $\odot(AXD)$ and Q be the antipode of B in $\odot(BXC)$. From $\angle PDA = \angle QCB = 90°$, we get that P and Q lie on CD. Moreover, from $\angle PXA = 90°$, we get that $P \in BX$, and similarly $Q \in AX$.

Being a diameter, $AP = 2 \cdot 13 = 26$, so by the Pythagorean theorem, $DP = \sqrt{26^2 - 24^2} = 10$. Similarly, $BQ = 30$ and $CQ = \sqrt{30^2 - 24^2} = 18$. Letting $AB = x$, we get $PQ = x - 28$. Quadrilateral $ABQP$ has perpendicular diagonals, so $AB^2 + PQ^2 = AP^2 + BQ^2$, which means that $x^2 + (x-28)^2 = 26^2 + 30^2$. Solving this quadratic gives $x = \boxed{14 + 4\sqrt{37}}$. (The condition that $\triangle AXD$ and $\triangle BXC$ are acute rules out $14 - 4\sqrt{37}$.)

10. A plane $\mathcal{P}$ intersects a rectangular prism at a hexagon which has side lengths 45, 66, 63, 55, 54, and 77, in that order. Compute the distance from the center of the rectangular prism to $\mathcal{P}$.

Proposed by: Albert Wang, Karthik Venkata Vedula

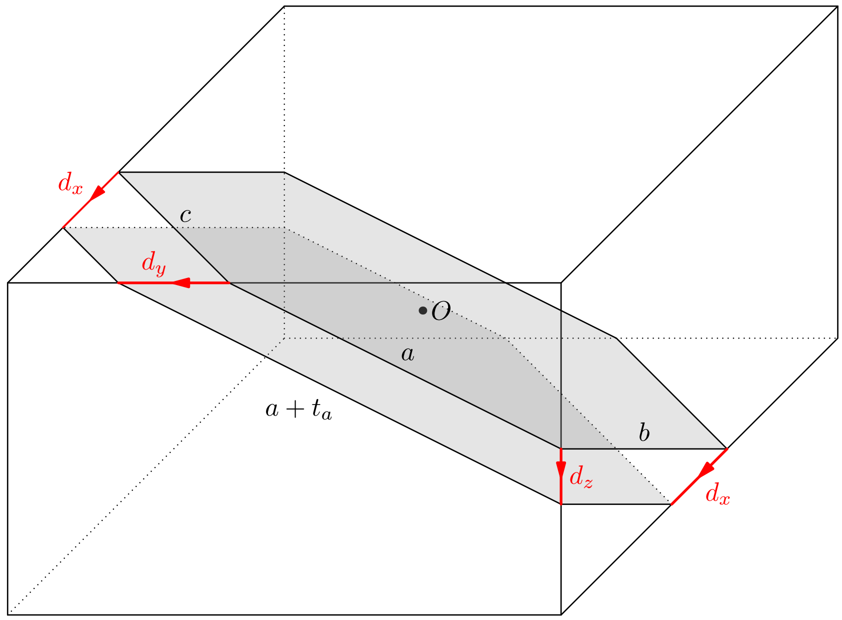
Answer: $\boxed{\sqrt{\frac{95}{24}}}$

Solution 1: Translate $\mathcal{P}$ so that it contains the center. The intersection of the translated plane with the rectangular prism is a centrally symmetric hexagon. Let its side lengths be a, b, c, a, b, and c, in that order. Then, for some t_a, t_b, and t_c, the side lengths of the hexagon before the translation were

$$(a - t_a, b + t_b, c - t_c, a + t_a, b - t_b, c + t_c) = (45, 66, 63, 55, 54, 77),$$

from which it follows that $t_a = 5$, $t_b = 6$, and $t_c = 7$.

Now, a translation of a plane can be written in one of three equivalent forms: it can be viewed as a translation in the x direction by a distance d_x, a translation in the y direction by a distance d_y, or a translation in the z direction by a distance d_z (with coordinate axes chosen as shown below).



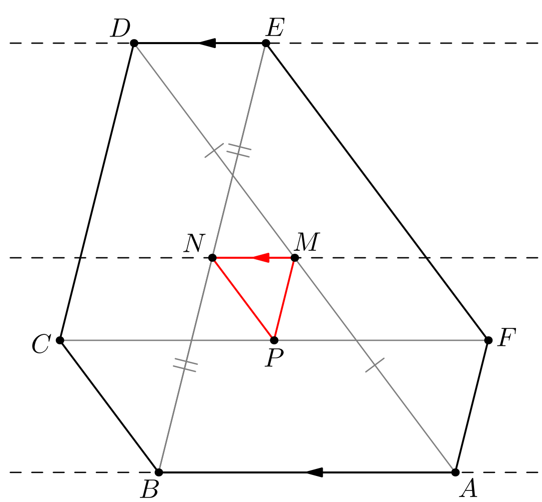
As shown above, we can express t_a, t_b, and t_c in terms of d_x, d_y, and d_z using the Pythagorean theorem, which yields $t_a = \sqrt{d_y^2 + d_z^2}$, $t_b = \sqrt{d_z^2 + d_x^2}$, and $t_c = \sqrt{d_x^2 + d_y^2}$. Hence,

$$(d_x^2, d_y^2, d_z^2) = \left(\frac{5^2 + 6^2 - 7^2}{2}, \frac{6^2 + 7^2 - 5^2}{2}, \frac{7^2 + 5^2 - 6^2}{2} \right) = (6, 30, 19).$$

We can draw a right pyramid with legs d_x, d_y, and d_z which has the center of the prism as a vertex with opposite face on $\mathcal{P}$. Then, the height of this pyramid, or the distance from the center to $\mathcal{P}$, is

$$\sqrt{\frac{1}{d_x^{-2} + d_y^{-2} + d_z^{-2}}} = \sqrt{\frac{1}{6^{-1} + 30^{-1} + 19^{-1}}} = \boxed{\sqrt{\frac{95}{24}}}.$$

Solution 2: Let the vertices of the hexagon be $ABCDEF$, where $AB = 45$, $BC = 66$, etc. Note that $AB \parallel DE$, $BC \parallel EF$, and $CD \parallel FA$. Let O be the center of the prism, and let M, N, and P be the midpoints of AD, BE, and CF, respectively.



The key observation is that MN is the midline between AB and DE. Hence, plane OMN is the midplane between the faces of the prism containing sides AB and DE. Similarly, planes OMP and ONP are the other two midplanes of the prism. Thus, OM, ON, and OP are mutually orthogonal.

Observe

$$MN = \frac{|AB - DE|}{2} = 5, \quad NP = \frac{|BC - EF|}{2} = 6, \quad \text{and} \quad PM = \frac{|CD - FA|}{2} = 7,$$

so by Heron's formula, we can compute the area of MNP to be $\sqrt{9(9-5)(9-6)(9-7)} = 6\sqrt{6}$.

Moreover, if $x = OM$, $y = ON$, and $z = OP$, then,

$$x^2 + y^2 = 5^2, \quad y^2 + z^2 = 6^2, \quad \text{and} \quad z^2 + x^2 = 7^2.$$

Solving this system of equations gives $x = \sqrt{19}$, $y = \sqrt{6}$, and $z = \sqrt{30}$. Therefore, if d is the distance from O to plane MNP (i.e., the answer), the volume of tetrahedron $OMNP$ can be written as

$$\tfrac{1}{6} \cdot \sqrt{19} \cdot \sqrt{6} \cdot \sqrt{30} = \tfrac{1}{3} \cdot (6\sqrt{6}) \cdot d,$$

so

$$d = \frac{\sqrt{19 \cdot 6 \cdot 30}}{2 \cdot 6\sqrt{6}} = \boxed{\sqrt{\frac{95}{24}}}.$$