HMMT February 2025

February 15, 2025

Team Round

1. **[20]** Let a, b, and c be pairwise distinct positive integers such that $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ is an increasing arithmetic sequence in that order. Prove that $\gcd(a, b) > 1$.

2. **[25]** A polyomino is a connected figure constructed by joining one or more unit squares edge-to-edge. Determine, with proof, the number of non-congruent polyominoes with no holes, perimeter 180, and area 2024.

3. **[30]** Let ω_1 and ω_2 be two circles intersecting at distinct points A and B. Point X varies along ω_1, and point Y on ω_2 is chosen such that AB bisects the angle $\angle XAY$. Prove that as X varies along ω_1, the circumcenter of $\triangle AXY$ (if it exists) varies along a fixed line.

4. **[35]** Jerry places at most one rook in each cell of a 2025×2025 grid of cells. A rook *attacks* another rook if the two rooks are in the same row or column and there are no other rooks between them.

   Determine, with proof, the maximum number of rooks Jerry can place on the grid such that no rook attacks 4 other rooks.

5. **[35]** Let $\triangle ABC$ be an acute triangle with orthocenter H. Points E and F are on segments $\overline{AC}$ and $\overline{AB}$, respectively, such that $\angle EHF = 90°$. Let X be the foot of the altitude from H to $\overline{EF}$. Prove that $\angle BXC = 90°$.

6. **[40]** Complex numbers $\omega_1, \ldots, \omega_n$ each have magnitude 1. Let z be a complex number distinct from $\omega_1, \ldots, \omega_n$ such that
$$\frac{z+\omega_1}{z-\omega_1} + \cdots + \frac{z+\omega_n}{z-\omega_n} = 0.$$
   Prove that $|z| = 1$.

7. **[45]** Determine, with proof, whether a square can be dissected into finitely many (not necessarily congruent) triangles, each of which has interior angles $30°$, $75°$, and $75°$.

8. **[50]** Let $\triangle ABC$ be a triangle with incenter I. The incircle of triangle $\triangle ABC$ touches $\overline{BC}$ at D. Let M be the midpoint of $\overline{BC}$, and let line AI meet the circumcircle of triangle $\triangle ABC$ again at $L \neq A$. Let ω be the circle centered at L tangent to AB and AC. If ω intersects segment $\overline{AD}$ at point P, prove that $\angle IPM = 90°$.

9. **[60]** Let $\mathbb{Z}$ be the set of integers. Determine, with proof, all primes p for which there exists a function $f\colon \mathbb{Z} \to \mathbb{Z}$ such that for any integer x,

   - $f(x+p) = f(x)$ and
   - p divides $f(x + f(x)) - x$.

10. **[60]** Determine, with proof, all possible values of $\gcd(a^2 + b^2 + c^2, abc)$ across all triples of positive integers (a, b, c).