HMMT HMIC 2025

April 20, 2025

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1. **[5]** Let $ABCD$ be a convex quadrilateral. Define parabolas $\mathcal{P}_A$, $\mathcal{P}_B$, $\mathcal{P}_C$, and $\mathcal{P}_D$ to have directrices BD, CA, DB, and AC, and foci A, B, C, and D, respectively. Prove that no two of these parabolas intersect more than once.

   (A parabola with directrix ℓ and focus P consists of all points X for which PX equals the distance from P to ℓ.)

2. **[7]** Find all polynomials P with real coefficients for which there exists a polynomial Q with real coefficients such that for all real t,
$$\cos(P(t)) = Q(\cos t).$$

3. **[8]** Let $ABCD$ be a parallelogram, and let O be a point inside $ABCD$. Suppose the circumcircles of triangles OAB and OCD intersect at $P \neq O$, and the circumcircles of triangles OBC and OAD intersect at $Q \neq O$. Prove $\angle POQ$ equals one of the angles of quadrilateral $ABCD$.

4. **[9]** Determine whether there exist infinitely many pairs of distinct positive integers m and n such that $2^m + n$ divides $2^n + m$.

5. **[13]** Compute the smallest positive integer $k > 45$ for which there exists a sequence a_1, a_2, a_3, $\ldots$, a_{k-1} of positive integers satisfying the following conditions:

   - $a_i = i$ for all integers $1 \leq i \leq 45$,
   - $a_{k-i} = i$ for all integers $1 \leq i \leq 45$, and
   - for any odd integer $1 \leq n \leq k-45$, the sequence a_n, a_{n+1}, $\ldots$, a_{n+44} is a permutation of $\{1, 2, \ldots, 45\}$.