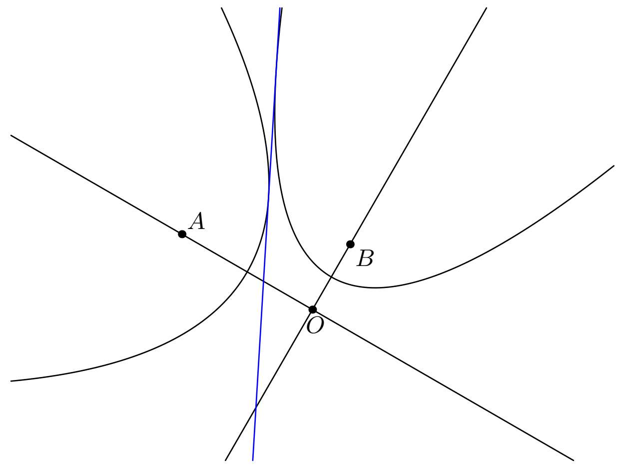
# HMIC 2025

**April 20-27, 2025**

1. **[5]** Let $ABCD$ be a convex quadrilateral. Define parabolas $\mathcal{P}_A$, $\mathcal{P}_B$, $\mathcal{P}_C$, and $\mathcal{P}_D$ to have directrices $BD$, $CA$, $DB$, and $AC$, and foci $A$, $B$, $C$, and $D$, respectively. Prove that no two of these parabolas intersect more than once.

   (A parabola with directrix $\ell$ and focus $P$ consists of all points $X$ for which $PX$ equals the distance from $P$ to $\ell$.)

   *Proposed by: Albert Wang*




**Solution 1:** Let $d(P, XY)$ be the distance from $P$ to line $XY$. We will first prove $\mathcal{P}_A$ and $\mathcal{P}_B$ intersect at most once.

**Claim 1.** Let $\ell_{AB}$ be the perpendicular bisector of $AB$. Then, $\mathcal{P}_A$ is tangent to $\ell_{AB}$.

*Proof.* Consider any point $X$ on $\mathcal{P}_A$. Then,

$$XA = d(X, BD) \leq XB,$$

with equality only holding at the unique point $X$ for which $XB \perp BD$. Thus, $\mathcal{P}_A$ lies entirely on one side of $\ell_{AB}$, touching it once at this point $X$. □

It follows that $\mathcal{P}_A$ and $\mathcal{P}_B$ are on different sides of $\ell_{AB}$ and hence can intersect at most once (possibly at a common tangency point to $\ell_{AB}$).

Since $ABCD$ is convex, $A$ and $C$ lie on opposite sides of line $BD$, the common directrix of $\mathcal{P}_A$ and $\mathcal{P}_C$. Thus, $\mathcal{P}_A$ and $\mathcal{P}_C$ lie on opposite sides of line $BD$ and cannot intersect at all.

The remaining pairs of parabolas are handled similarly.

**Solution 2:** Let $d(P, XY)$ be the distance from $P$ to line $XY$.

**Claim 2.** $\mathcal{P}_A$ and $\mathcal{P}_B$ intersect at most once.

*Proof.* Let $P$ be a common point of both parabolas. Then, $d(P, BD) = PA$ and $d(P, AC) = PB$, which combined imply

$$PA = d(P, BD) \leq PB = d(P, AC) \leq PA.$$

Thus, the inequalities above are equalities, i.e., $PA = PB$, $PA \perp AC$, and $PB \perp BD$. Such $P$, if it exists, is unique. □

The remaining pairs of parabolas are handled similarly. As in solution 1, $\mathcal{P}_A$ and $\mathcal{P}_C$ cannot intersect, nor can $\mathcal{P}_B$ and $\mathcal{P}_D$.

2. **[7]** Find all polynomials $P$ with real coefficients for which there exists a polynomial $Q$ with real coefficients such that for all real $t$,

$$\cos(P(t)) = Q(\cos t).$$

*Proposed by: Karthik Venkata Vedula*

**Answer:** $\boxed{\text{All constant functions and } P(x) = ax + b\pi \text{ for all nonzero integers } a \text{ and integers } b}$

**Solution 1:** It is well-known that these polynomials work by taking $Q$ to be a Chebyshev polynomial (if $P$ is linear) or a constant (if $P$ is constant).

Suppose that $\deg P \geq 2$. Now consider the density of the roots of $\cos(P(t))$, i.e.

$$\lim_{n\to\infty} \frac{\text{number of roots in the interval } [-n, n]}{n}.$$

Since $\cos(P(t)) = Q(\cos t)$, the density is finite, because for each interval of length $2\pi$, there can only be a finite number of roots (i.e. twice the degree of $Q$). However, we claim that $\cos(P(t))$ has an infinite density of roots. In particular, consider the solutions to $P(x) = \pm(2k-1)\pi/2$ over positive integers $k$. Asymptotically, for large $k$, such $x$ will always exist and be $\Theta(k^{1/\deg P})$. As $k \to \infty$, such $x$ become infinitely dense, contradicting the finite density of roots of $Q(\cos t)$. Therefore, $\deg P \leq 1$.

If $\deg P = 1$, let $P(t) = at + b$. Observe that $Q(\cos t)$ is periodic with period $2\pi$, so $\cos(P(t))$ must be as well. This is only the case when $a$ is an integer. Furthermore,

$$Q(\cos t) = \cos(at + b) = \cos(at)\cos(b) - \sin(at)\sin(b)$$

must be an even function. Note that $\cos(at)\cos(b)$ is even and $\sin(at)\sin(b)$ is odd, so $\sin(at)\sin(b) = 0$ for all $t$, and hence $b$ is an integer multiple of $\pi$.

Thus, the only polynomials that work are the ones claimed above.

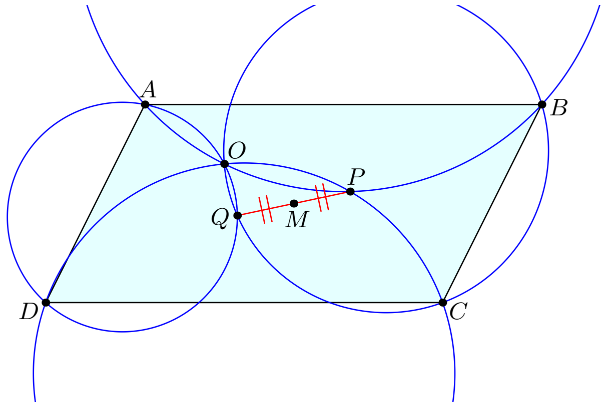
**Solution 2:** Taking the derivative of both sides,

$$\sin(P(t))P'(t) = (-\sin t)Q'(\cos t).$$

The right-hand side is bounded in $t$, so the left-hand side must also be bounded. If $\deg P \geq 2$, then as $t$ approaches $\infty$, $P'(t)$ approaches $\pm\infty$ and $\sin(P(t))$ does not approach 0, contradiction. Thus, $\deg P \leq 1$, and we finish as before.

3. **[8]** Let $ABCD$ be a parallelogram, and let $O$ be a point inside $ABCD$. Suppose the circumcircles of triangles $OAB$ and $OCD$ intersect at $P \neq O$, and the circumcircles of triangles $OBC$ and $OAD$ intersect at $Q \neq O$. Prove $\angle POQ$ equals one of the angles of quadrilateral $ABCD$.

*Proposed by: Derek Liu*




**Solution:** In what follows, all angles are directed.

**Claim 1.** The points $P$ and $Q$ are symmetric over the center of $ABCD$.

*Proof.* Note that

$$\measuredangle APB = \measuredangle AOB = \measuredangle OAD + \measuredangle CBO = \measuredangle OQD + \measuredangle CQO = \measuredangle CQD.$$

Similar equalities hold for each pair of opposite sides, so $P$ and $Q$ are symmetric across the parallelogram's center. $\square$

Consequently, $AP$ and $CQ$ are parallel, so

$$\measuredangle POQ = \measuredangle POB + \measuredangle BOQ = \measuredangle PAB + \measuredangle BCQ = \measuredangle CBA,$$

as desired. (Once we undirect the angles, $\angle POQ$ is either $\angle B$ or $\pi - \angle B = \angle A$.)

4. **[9]** Determine whether there exist infinitely many pairs of distinct positive integers $m$ and $n$ such that $2^m + n$ divides $2^n + m$.

*Proposed by: Carlos Rodriguez, Jordan Lefkowitz*

**Answer:** $\boxed{\text{Yes}}$

**Solution:** Let $k$ be a positive integer, and set $m = 2^k$ and $n = p - 2^{2^k}$ for prime $p$ to be chosen later. We want $2^m + n = p$ to divide $2^{p-2^{2^k}} + 2^k$, which is equivalent to having

$$0 \equiv 2^{p-2^{2^k}} + 2^k \equiv 2^{1-2^{2^k}} + 2^k \equiv 2^{1-2^{2^k}}\left(2^{2^{2^k}+k-1} + 1\right) \pmod{p}.$$

Let $r = 2^{2^k} + k - 1$. Since $r \neq 3$, by Zsigmondy, we can pick a prime $p$ that divides $2^r + 1$ but not $2^s + 1$ for any nonnegative integer $s < r$. Let $d = \operatorname{ord}_p(2)$. Then, $2^{|d-r|} \equiv -1 \pmod{p}$, so by definition of $p$, we have $|d - r| \geq r$. Hence, $d \geq 2r$. As $d \mid p - 1$, we conclude

$$p > 2r = 2\left(2^{2^k} + k - 1\right) > 2^{2^k} + 2^k,$$

so $n = p - 2^{2^k} > m$. Since $p \mid 2^r + 1$ by definition, $(m, n)$ is a pair of distinct positive integers with $2^m + n \mid 2^n + m$.

As $k$ was arbitrary (and $m = 2^k$), there exist infinitely many such pairs.

5. **[13]** Compute the smallest positive integer $k > 45$ for which there exists a sequence $a_1$, $a_2$, $a_3$, $\ldots$, $a_{k-1}$ of positive integers satisfying the following conditions:

- $a_i = i$ for all integers $1 \le i \le 45$,
- $a_{k-i} = i$ for all integers $1 \le i \le 45$, and
- for any odd integer $1 \le n \le k-45$, the sequence $a_n$, $a_{n+1}$, $\ldots$, $a_{n+44}$ is a permutation of $\{1, 2, \ldots, 45\}$.

*Proposed by: Derek Liu*

**Answer:** $\boxed{1059}$

**Solution:** First, we show 1059 is optimal. Assume for sake of contradiction that $k < 1059$.

The given condition ensures that $\{a_1, a_2\} = \{a_{46}, a_{47}\}$, $\{a_3, a_4\} = \{a_{48}, a_{49}\}$, and so on. In particular, if $a_i = j$, the next appearance of $j$ must either be $a_{i+44}$, $a_{i+45}$, or $a_{i+46}$; working modulo 45, these indices all differ from $i$ by at most 1. Furthermore, $a_i = a_{i+44}$ is only possible if $i$ is even, and $a_i = a_{i+46}$ is only possible if $i$ is odd.

Also, $a_1 \ne a_{45}$ by definition. Thus, if the sequence contains the same number at least 25 times, the 25th appearance has index at least $2 + 24 \cdot 44 = 1058$, implying $k \ge 1059$. Hence, we can assume every number appears at most 24 times in the sequence.

Observe that there exists a unique integer $1 \le j \le 45$ such that

$$(k-j) - j = k - 2j \equiv 22 \pmod{45}.$$

Let $k_1 = j$, $k_2$, $k_3$, $\ldots$, $k_\ell = k - j$ be the indices of where $j$ appears in the sequence; as assumed above, $\ell \le 24$. For any $i$, we proved $k_i - k_{i-1}$ is either 44, 45, or 46, and thus either 0 or $\pm 1$ modulo 45. Since $k - j$ and $j$ differ by $22 \equiv -23$ modulo 45, either $k_i - k_{i-1} = 46$ for at least 22 different $i$, or $k_i - k_{i-1} = 44$ for at least 23 different $i$. We split into cases based on which.

If $k_i - k_{i-1} = 46$ at least 22 times, we claim $\ell = 23$. Indeed, if $\ell = 24$, then

$$(k-j) \ge j + 22 \cdot 46 + 44 = j + 1056.$$

Since $(k-j) - j \equiv 22 \pmod{45}$, we actually have $(k-j) \ge j + 1057$, so $k \ge 2j + 1057 \ge 1059$, contradiction.

Thus, $\ell = 23$, so $k_i - k_{i-1} = 46$ for all $i$, and $k = 2j + 22 \cdot 46 = 2j + 1012$, which means we can assume $j \le 23$ (otherwise $k > 1059$).

Since $a_j = a_{j+46}$, we must also have $a_{j+1} = a_{j+45}$ be the second appearance of $j+1$, which means $j$ must be odd. Then, as $j + 45$ is even, $a_{j+45} = j + 1$ must either be equal to either $a_{j+89}$ or $a_{j+90}$. Now, $a_{k-j-1} = j + 1$ as well, and

$$(k-j-1) - (j+90) \equiv 21 \pmod{45},$$

so if $a_{j+90}$ is the 3rd appearance of $j+1$, then $a_{k-j-1}$ must be at least the 24th. Thus, it must be exactly the 24th, with each appearance of $j+1$ being exactly 46 terms after the previous. Thus,

$$k - j - 1 \ge (j+90) + 21 \cdot 46 = j + 1056,$$

and $k \ge 2j + 1057 \ge 1059$, as desired. If $a_{j+89}$ is the 3rd appearance of $j+1$, then since $(k-j-1) - (j+89) \equiv 22 \pmod{45}$, it follows that $a_{k-j-1}$ must be at least the 25th appearance, contradiction.

The other case, where $k_i - k_{i-1} = 44$ at least 23 times (i.e., for all $i$) plays out similarly. Since $k_2 = k_1 + 44$, we immediately have $j = k_1$ is even. Furthermore, $a_{j-1} = a_{j+45} = j - 1$ is the second appearance of $j-1$, and since $j + 45$ is odd, the third appearance is either $a_{j+90}$ or $a_{j+91}$. However,

$$(k-j+1) - (j+90) \equiv -22 \pmod{45},$$

so regardless of whether $a_{j+90}$ or $a_{j+91}$ is the 3rd appearance of $j-1$, we know $a_{k-j+1}$ must be at least the 25th, contradiction.

Thus $k \geq 1059$, and it suffices to provide a construction.

Given an integer $m$, let $\alpha$ represent the permutation on $\{1, 2, \ldots, m\}$ given by swapping 1 and 2, 3 and 4, etc. and let $\beta$ represent the permutation given by swapping 2 and 3, 4 and 5, etc.

**Claim 1.** Both of the products $\underbrace{\cdots\beta\alpha\beta\alpha}_{m}$ and $\underbrace{\cdots\alpha\beta\alpha\beta}_{m}$, with exactly $m$ terms in the products, equal the reverse permutation.

*Proof.* If we track any number $i$, observe that if $i$ is odd,

$$\begin{aligned}
\alpha(i) &= i+1 \\
\beta\alpha(i) &= i+2 \\
&\vdots \\
\underbrace{\cdots\beta\alpha}_{m-i}(i) &= m \\
\underbrace{\cdots\beta\alpha}_{m-i+1}(i) &= m \\
\underbrace{\cdots\beta\alpha}_{m-i+2}(i) &= m-1 \\
&\cdots \\
\underbrace{\cdots\beta\alpha}_{m}(i) &= m-i+1.
\end{aligned}$$

A similar chain occurs if $i$ is even, so $\underbrace{\cdots\beta\alpha}_{m}$ is indeed the reverse permutation. The same argument shows that the product $\underbrace{\cdots\alpha\beta}_{m}$ is also the reverse permutation. $\square$

Our construction now goes as follows: for any odd $n$, let $a_{n+46} = a_n$ and $a_{n+45} = a_{n+i}$ unless $n \equiv 0$ or $23 \bmod 45$. Then, we can note that $a_{46}$ through $a_{68}$ are the permutation $\alpha$ on $\{1, 2, \ldots, 23\}$, and $a_{91}$ through $a_{113}$ are $\alpha\beta$, etc. By our claim, $a_{1+23\cdot 45} = a_{1036}$ through $a_{1058}$ are 1 through 23 reversed. Similarly, as there are only 22 numbers from 24 through 45, we know $a_{24+22\cdot 45} = a_{1014}$ through $a_{1035}$ are 24 through 45 reversed. It follows that this sequence satisfies the above conditions for $k = \boxed{1059}$.

The diagram below shows what the construction would look like when 45 is replaced with 9. The sequence can be read row-by-row. The vertical line separates 1 through 5 from 6 through 9.

$$\begin{array}{ccccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 1 & 4 & 3 & 5 & 6 & 8 & 7 & 9 \\
2 & 4 & 1 & 5 & 3 & 8 & 6 & 9 & 7 \\
4 & 2 & 5 & 1 & 3 & 8 & 9 & 6 & 7 \\
4 & 5 & 2 & 3 & 1 & 9 & 8 & 7 & 6 \\
5 & 4 & 3 & 2 & 1 & & & &
\end{array}$$

*Remark.* The exact same proof and construction work for all odd $n \geq 5$, yielding an answer of $\frac{n^2+2n+3}{2}$. Notably, for $n = 3$ the answer is 6 instead of 9.