HMMT November 2025

November 08, 2025

Guts Round

1. **[5]** Compute the remainder when 1000! is divided by 1001.

   Proposed by: Derek Liu

   Answer: $\boxed{0}$

   Solution: Since 1001 can be factored into $7 \cdot 11 \cdot 13$, we know 1000! is a multiple of 1001, so the remainder is $\boxed{0}$.

2. **[5]** Sebastian fills the four squares in $\square^{\square^{\square^{\square}}}$ with the numbers 2, 0, 2, and 5, in some order. Compute the sum of all distinct possible values of the resulting expression.

   (Here, $a^{b^{c^d}}$ is evaluated as $a^{(b^{(c^d)})}$. For instance, $2^{2^{2^2}} = 2^{2^4} = 2^{16} = 65536$.)

   Proposed by: Sebastian Attlan

   Answer: $\boxed{69}$

   Solution: We do casework on the location of the 0. Rewrite the expression as $a^{b^{c^d}}$.

   - If $a = 0$, the expression is simply 0.
   - If $b = 0$, then the expression must be $a^0 = 1$.
   - If $c = 0$, then
   $$a^{b^{c^d}} = a^{b^0} = a^1 = a.$$
   There are 2 distinct values for a, being 2 or 5.
   - If $d = 0$, then
   $$a^{b^{c^d}} = a^{b^1} = a^b,$$
   which can take on the values $2^2 = 4$, $2^5 = 32$, and $5^2 = 25$.

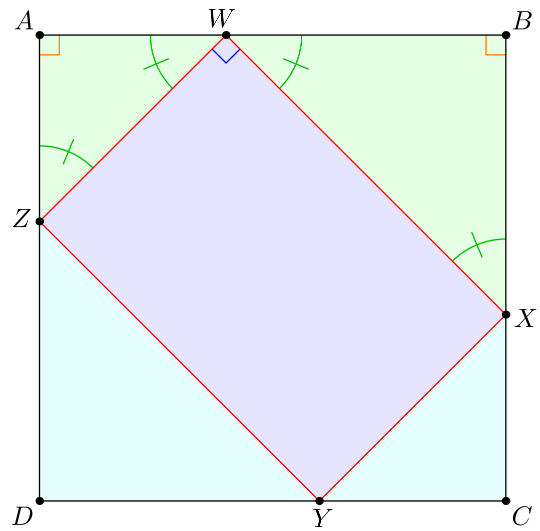
   Our answer is thus $(0) + (1) + (2 + 5) + (4 + 25 + 32) = \boxed{69}$.

3. **[5]** Square $ABCD$ has side length 45. Points W, X, Y, and Z lie on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, such that $AW = CY = 20$ and $BX = DZ = 25$. Compute the area of quadrilateral $WXYZ$.

   Proposed by: Derek Liu

   Answer: $\boxed{1000}$

   Solution:



Note that $AW = AZ = 20$ and $BW = BX = 25$. Therefore, triangles AWZ and BXW are isosceles right triangles, whence $WZ = 20\sqrt{2}$ and $WX = 25\sqrt{2}$. Both WX and WZ make 45-degree angles with AB, so we get that WZ is perpendicular to WX. Similarly, WX is perpendicular to XY and WZ is perpendicular to ZY. Hence, $WXYZ$ is a rectangle. Thus, the area of $WXYZ$ is

$$WZ \cdot WX = (20\sqrt{2}) \cdot (25\sqrt{2}) = \boxed{1000}.$$

4. **[6]** A mercury thermometer reads the temperature using three temperature scales: $^\circ C$ (Celsius), $^\circ F$ (Fahrenheit), and $^\circ S$ (Saengrungkongka). The conversions are as follows:

- A temperature of $x^\circ C$ corresponds to $(\frac{9}{5}x + 32)^\circ F$.
- A temperature of $x^\circ S$ corresponds to $(\frac{9}{5}x + 32)^\circ C$.

Given that the current temperature readings on all three scales are odd positive integers, compute the minimum possible value of the current temperature in degrees **Celsius**.

Proposed by: Jackson Dryg

Answer: $\boxed{95}$

Solution: Let the current temperature readings be f, c, and s in degrees Fahrenheit, Celsius, and Saengrungkongka, respectively. Write f and s in terms of c:

$$f = \frac{9}{5}c + 32 \quad \text{and} \quad s = \frac{5}{9}(c - 32).$$

We are given that f, c, and s are odd positive integers. For f to be an integer, c must be a multiple of 5. For s to be an integer, $c - 32$ must be a multiple of 9. Since c must also be odd, we get that

$$\begin{aligned} c &\equiv 0 \pmod{5} \\ c &\equiv 32 \equiv 5 \pmod{9} \\ c &\equiv 1 \pmod{2} \end{aligned}$$

Note that $c = 5$ is a solution to this system of congruences. The least common multiple of the moduli 2, 5, and 9 is 90, so by the Chinese Remainder Theorem, all solutions are given by $c \equiv 5 \pmod{90}$. As c is positive, the smallest values we need to check are 5 and 95.

Observe that $c = 5$ fails since that leads to $s = -15 < 0$. On the other hand, $c = 95$ works, since the corresponding values are $s = 35$ and $f = 203$, which are odd positive integers. Hence, the minimum possible value of the current temperature in degrees Celsius is $\boxed{95}$.

5. **[6]** A circle of radius strictly less than 2 is drawn in the plane. Compute the maximum possible number of lattice points that can lie on its circumference. (A lattice point is a point with integer coordinates.)

Proposed by: Jordan Lefkowitz

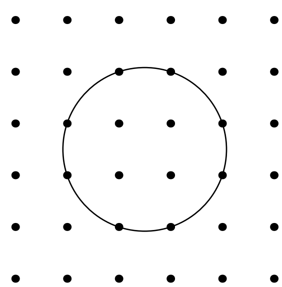
Answer: $\boxed{8}$

Solution:

Observe that every lattice point must lie on a horizontal grid line (i.e., a line $y = k$ where k is an integer). Then note the following:

- The circle must intersect at most 4 horizontal grid lines. This is because for any set of 5 horizontal grid lines, the distance between the topmost and bottommost lines is at least 4. But the maximum distance between any two points on the circle is its diameter, which is less than 4, so it cannot intersect all 5 lines.
- Each horizontal grid line can lead to at most 2 lattice point intersections, since any circle intersects a line at most twice.

Thus, the number of lattice point intersections is at most $4 \cdot 2 = \boxed{8}$. For a construction, consider the circle $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{2}$, which intersects 8 lattice points as shown below.

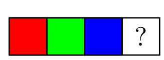


6. **[6]** Compute the number of ways to color each cell of an 8×8 grid either red, green, or blue such that every 1×3 and 3×1 rectangle with edges on the grid lines contains exactly one cell of each color.

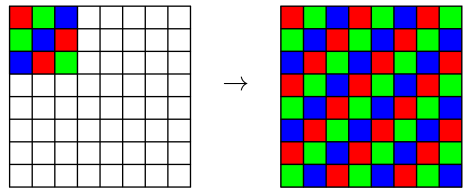
Proposed by: Jackson Dryg

Answer: $\boxed{12}$

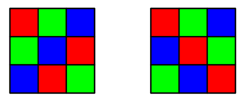
Solution: Notice that two squares 3 units apart must be the same color. For example, the cell marked ? must be red.



If we color only the top-left 3×3 square, by using this fact repeatedly, there's exactly one way to color the rest of the grid:



So we only need to count the ways to color the top-left 3×3 square. There are $3 \times 2 \times 1 = 6$ ways to choose the colors on the top row. Consider one of these options, say red-green-blue. Each row and column must contain exactly one red cell, so there are exactly two ways to place the other two red cells. Both of these lead to exactly one way to color the entire 3×3 square:



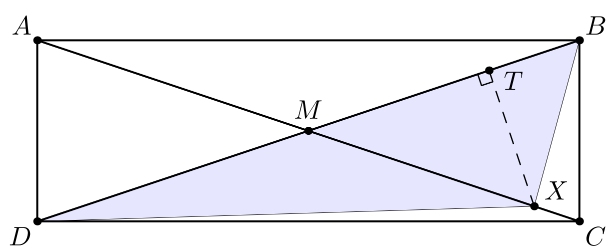
Similarly, for each of the 6 ways to color the top row, there are 2 ways to color the rest of the 3×3 square. Therefore, the answer is $6 \times 2 = \boxed{12}$.

7. **[7]** Point X lies on diagonal $\overline{AC}$ of rectangle $ABCD$ such that $AX = 11$, $CX = 1$, and triangle BXD has area 18. Given that $BX < DX$, compute BX.

Proposed by: Jason Mao

Answer: $\boxed{\sqrt{13}}$

Solution: Let $\overline{AC}$ and $\overline{BD}$ meet at M, and let T be the foot of the altitude from X onto $\overline{BD}$.



Diagonals $\overline{AC}$ and $\overline{BD}$ both have length $AX + CX = 12$. Since $\triangle BXD$ has area 18, we have:

$$\frac{1}{2}(BD)(TX) = 18 \implies TX = \frac{18 \cdot 2}{12} = 3.$$

Furthermore, $MX = MC - CX = 6 - 1 = 5$. Then the Pythagorean Theorem on $\triangle MTX$ yields:

$$MT = \sqrt{MX^2 - TX^2} = \sqrt{5^2 - 3^2} = 4.$$

Finally, $BT = MB - MT = 6 - 4 = 2$, so the Pythagorean Theorem on $\triangle BTX$ yields:

$$BX = \sqrt{TB^2 + TX^2} = \sqrt{2^2 + 3^2} = \boxed{\sqrt{13}}.$$

8. **[7]** Compute the sum of the distinct prime factors of 20202525.

Proposed by: Sebastian Attlan

Answer: $\boxed{243}$

Solution: Observe that

$$\begin{aligned} 20202525 &= 5 \cdot 4040505 \\ &= 5^2 \cdot 808101 \\ &= 5^2 \cdot 101 \cdot 8001. \end{aligned}$$

Furthermore,

$$8001 = 20^3 + 1^3 = (20+1)(20^2 - 20 \cdot 1 + 1^2) = 21 \cdot 381 = 3^2 \cdot 7 \cdot 127.$$

Since 127 is prime, the answer is $3 + 5 + 7 + 101 + 127 = \boxed{243}$.

9. **[7]** Suppose S and T are two sets of distinct positive integers, each with 15 elements, such that S and T have no elements in common. Further suppose

$$\text{sum}(S) = \text{sum}(T) = k,$$

where $\text{sum}(A)$ denotes the sum of the elements of A. Compute the minimum possible value of k.

Proposed by: Rishabh Das

Answer: $\boxed{233}$

Solution: The answer is $\boxed{233}$. This is achieved by the construction

$$S = \{1, 2, 3, 4, 5, 6, 8, 22, 23, 24, 25, 26, 27, 28, 29\}$$

and

$$T = \{7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 31\}.$$

To show that $k = 233$ is optimal, note that

$$2k = \text{sum}(S) + \text{sum}(T) \geq \sum_{i=1}^{30} i = 465.$$

This implies $k \geq 233$, as desired.

10. **[8]** Compute the number of positive divisors of 10^{20} that leave a remainder of 1 when divided by 9.

Proposed by: Rohan Bodke

Answer: $\boxed{75}$

Solution: Note that all divisors of 10^{20} can be written as $2^a 5^b$, where a and b are nonnegative integers with $0 \leq a \leq 20$, $0 \leq b \leq 20$. For this divisor to leave a remainder of 1 when divided by 9, we need

$$\begin{aligned} & 2^a 5^b \equiv 1 \pmod 9 \\ \implies & 2^a \equiv (5^{-1})^b \pmod 9 \\ \implies & 2^a \equiv 2^b \pmod 9 \\ \implies & 2^{a-b} \equiv 1 \pmod 9 \\ \implies & a \equiv b \pmod 6. \end{aligned}$$

For $a \equiv b \equiv 0, 1, 2 \pmod 6$, there are four possible values of a and four possible values of b. For $a \equiv b \equiv 3, 4, 5 \pmod 6$, there are three possible values of a and three possible values of b. Thus, the answer is $3(4 \cdot 4 + 3 \cdot 3) = \boxed{75}$.

11. **[8]** Jessica has a non-square rectangular sheet of paper with all 4 corners colored differently. She repeats the following process 8 times: she picks one of the rectangle's two axes of symmetry, then flips the rectangle over that axis. Compute the number of ways she can do this so that each corner ends up in its original position.

Proposed by: Kira Lewis

Answer: $\boxed{128}$

Solution 1: Let the corners be A, B, C and D in that order around the rectangle. Each time the sheet is flipped, the side of the sheet facing upward changes. As there are an even number of flips, the side of the sheet facing upward is the same as the original side. Hence, if A ends up in its original corner, then all the other corners will also end up in their original places.

During each flip, A will move to one of the corners adjacent to it previous position. In 7 flips, A will end up in the original position of either B or D. After this, there is always a unique way to flip the rectangle so that A ends up in its original position. Thus, there are 2 choices for each of the first seven flips, and the last flip is uniquely determined. This leads to a total of 2^7 ways to flip the rectangle so that it ends up in the original position, and so the answer is $\boxed{128}$.

Solution 2: Label two corners of the rectangle on the left "L" and the other two corners on the right "R". Label the horizontal axis as x, vertical axis as y. When the flipping is over y axis, L-corners change the side from left to right or from right to left. When the flipping is over x axis, the L-corners do not change left-right side.

Therefore, for each corner to end up in its original position, the axis y should be chosen even times, so the axis x must be chosen even times too.

On the other hand, suppose that each of axis x and y is chosen for even times. Applying the same trick to each axis x, y separately, we can conclude that each corner ends up in the same side with respect to both axes. (that is, if a corner starts above x axis, it ends up above x axis. If it starts on the left to y axis, it ends up to the left of y axis). Consequently, each corner ends up at the start position.

As a result, the given condition in the problem is true if and only if axes x, y are each selected for even numbers of time. The number of all such combinations is

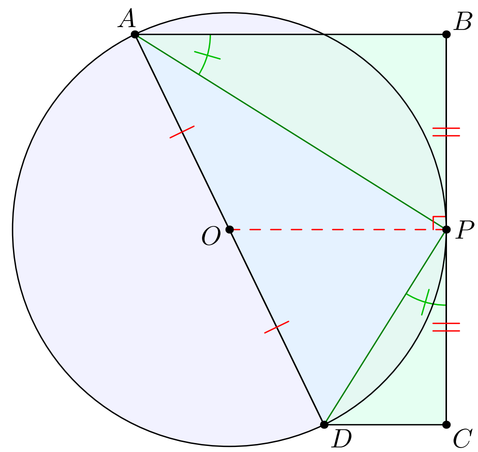
$$\binom{8}{8} + \binom{8}{6} + \binom{8}{4} + \binom{8}{2} + \binom{8}{0} = \boxed{128}.$$

12. **[8]** Let $ABCD$ be a right trapezoid such that $\angle ABC = \angle BCD = 90^\circ$ and the circle with diameter $\overline{AD}$ is tangent to side $\overline{BC}$. Given that $AB = 7$ and $BC = 8$, compute CD.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{16}{7}}$

Solution:



Note that the center O of the circle with diameter $\overline{AD}$ is the midpoint of $\overline{AD}$. Let P denote the tangency point of $\overline{BC}$. Because $\overline{OP}$ is perpendicular to $\overline{BC}$, we must have that P is the midpoint of $\overline{BC}$, implying $BP = PC = 4$. Next, notice that because $\angle ABP = \angle PCD = 90^\circ$ and $\angle BPA + \angle DPC = 90^\circ$, we have $\triangle ABP \sim \triangle PCD$, so

$$\frac{AB}{BP} = \frac{PC}{CD} \implies CD = \frac{BP \cdot PC}{AB} = \boxed{\frac{16}{7}}.$$

13. **[9]** Let P be a point and ℓ be a line in the coordinate plane.

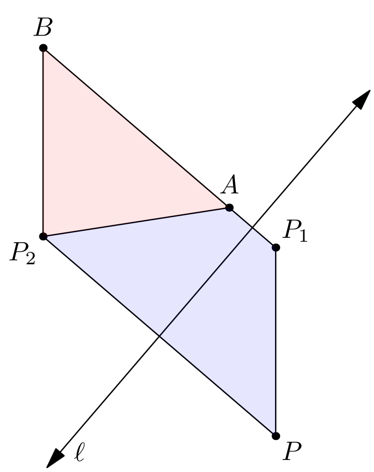
- If point P were reflected across ℓ and then translated by $(+0, +6)$, the result would be point A.
- If point P were translated by $(+0, +6)$ and then reflected across ℓ, the result would be point B.

Given that $AB = 10$, compute the maximum possible area of triangle PAB.

Proposed by: Jason Mao

Answer: $\boxed{5\sqrt{11}}$

Solution: Let P_1 be the translation of P by $(+0, +6)$, and let P_2 be the reflection of P about ℓ.



Lemma 1. We have $[PAB] = [P_2AB]$.

Proof. First note that PP_1AP_2 is an isosceles trapezoid, as segment $\overline{AP_2}$ is the reflection of segment $\overline{P_1P}$ over ℓ. Also, PP_1BP_2 is a parallelogram, as P_1 and B are 6 units above P and P_2, respectively. Thus, both P_1A and P_1B are parallel to PP_2, so $PP_2 \parallel AB$.

Thus, the heights from P and P_2 onto side $\overline{AB}$ have the same length, so $\triangle PAB$ and $\triangle P_2AB$ have heights and bases of the same length, meaning their areas must be the same too. □

We now compute the area of $\triangle P_2AB$ as follows:

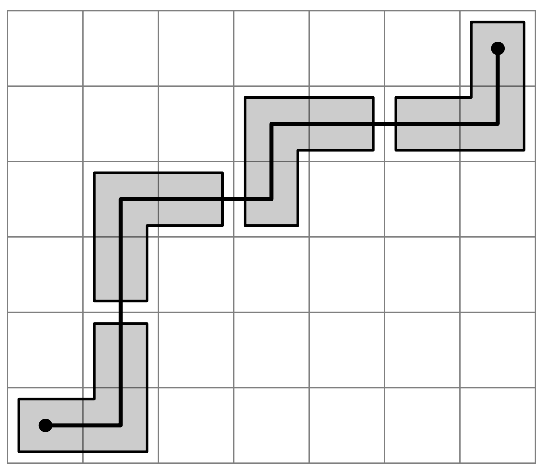
- Since P_2 and A are reflections of P and P_1 about ℓ, respectively, we have $P_2A = PP_1 = 6$.
- Also, we are given $P_2B = 6$ and $AB = 10$.

Thus, $\triangle P_2AB$ is isosceles with base length 10 and height $\sqrt{6^2 - 5^2} = \sqrt{11}$, so

$$[PAB] = [P_2AB] = \frac{1}{2} \cdot 10 \cdot \sqrt{11} = \boxed{5\sqrt{11}}.$$

14. **[9]** Marin starts on the bottom-left square of a 6×7 grid and walks to the top-right square by taking steps one square either up or to the right. Given that the set of squares Marin visits on his walk can be partitioned into L-trominoes, compute the number of ways that Marin can complete his walk.

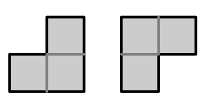
An *L-tromino* is a set of three squares formed by removing exactly one square from a 2×2 grid of squares. One example of a valid path is shown below:



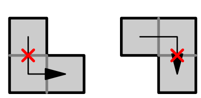
Proposed by: Andrew Brahms

Answer: $\boxed{48}$

Solution: Note first that each of the L-trominoes must be oriented in one of the following two pictured configurations:



This is because each of the other two types of L-tromino would induce either a leftward or downward movement, as shown below, which is forbidden by the problem statement.



Additionally note that because the grid has 7 columns and 6 rows, Marin will move right 6 times and up 5 times to get to the top-right corner, passing through a total of 12 squares. This additionally implies that we will use a total of 4 L-trominoes when tiling his path.

Each L-tromino will correspond to Marin moving one unit to the right and one unit up, and we have either an up or a right move connecting every pair of consecutive L-trominoes along Marin's path. Because Marin moves to the right a total of 6 times and up a total of 5, and because one of each move type occurs within each of the 4 L-trominoes, we therefore have a total of $6-4=2$ right moves and $5-4=1$ up move between pairs of consecutive L-trominoes.

Since each of our 4 L-trominoes can be positioned in one of 2 ways, and there are 3 ways to arrange the up and right moves between consecutive L-trominoes, we thus have a total of $2^4 \cdot 3 = \boxed{48}$ possible paths.

15. **[9]** Compute

$$\sum_{k=1}^{\infty} \frac{1}{2^{2^k} - 2^{-2^k}} = \frac{1}{2^{2^1} - 2^{-2^1}} + \frac{1}{2^{2^2} - 2^{-2^2}} + \frac{1}{2^{2^3} - 2^{-2^3}} + \cdots .$$

Proposed by: Jacopo Rizzo

Answer: $\boxed{\frac{1}{3}}$

Solution: Note that for all real numbers a,

$$\frac{1}{a - a^{-1}} = \frac{1}{a-1} - \frac{1}{a^2-1}.$$

Plugging in $a = 2^{2^n}$, we obtain

$$\frac{1}{2^{2^n} - 2^{-2^n}} = \frac{1}{2^{2^n} - 1} - \frac{1}{2^{2^{n+1}} - 1}.$$

Thus, the n-th partial sum is

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{2^{2^k} - 2^{-2^k}} &= \left(\frac{1}{2^{2^1}-1} - \frac{1}{2^{2^2}-1}\right) + \left(\frac{1}{2^{2^2}-1} - \frac{1}{2^{2^3}-1}\right) + \cdots + \left(\frac{1}{2^{2^n}-1} - \frac{1}{2^{2^{n+1}}-1}\right) \\ &= \frac{1}{2^{2^1}-1} - \frac{1}{2^{2^{n+1}}-1}.\end{aligned}$$

As $n \to \infty$, we have $1/(2^{2^{n+1}} - 1) \to 0$, which implies that

$$\sum_{k=1}^{\infty} \frac{1}{2^{2^k} - 2^{-2^k}} = \frac{1}{2^{2^1}-1} = \boxed{\frac{1}{3}}.$$

16. **[10]** Let a_1, a_2, a_3, a_4, and a_5 be the five distinct complex solutions of $x^5 - 20x + 25 = 0$. Compute $a_1^4 + a_2^4 + a_3^4 + a_4^4 + a_5^4$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{80}$

Solution: Note that since $a_i^5-20a_i+25=0$, we have $a_i^4=20-\frac{25}{a_i}$ as 0 is not a root of $x^5-20x+25=0$. Therefore,

$$\begin{aligned}
a_1^4+a_2^4+a_3^4+a_4^4+a_5^4 &= \left(20-\frac{25}{a_1}\right)+\left(20-\frac{25}{a_2}\right)+\left(20-\frac{25}{a_3}\right)+\left(20-\frac{25}{a_4}\right)+\left(20-\frac{25}{a_5}\right)\\
&= 100-25\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5}\right)\\
&= 100-25\cdot\frac{a_2a_3a_4a_5+a_1a_3a_4a_5+a_1a_2a_4a_5+a_1a_2a_3a_5+a_1a_2a_3a_4}{a_1a_2a_3a_4a_5}
\end{aligned}$$

By Vieta's formulas,

$$\begin{aligned}
a_2a_3a_4a_5+a_1a_3a_4a_5+a_1a_2a_4a_5+a_1a_2a_3a_5+a_1a_2a_3a_4 &= -20\\
a_1a_2a_3a_4a_5 &= -25
\end{aligned}$$

Substituting these values in, we get

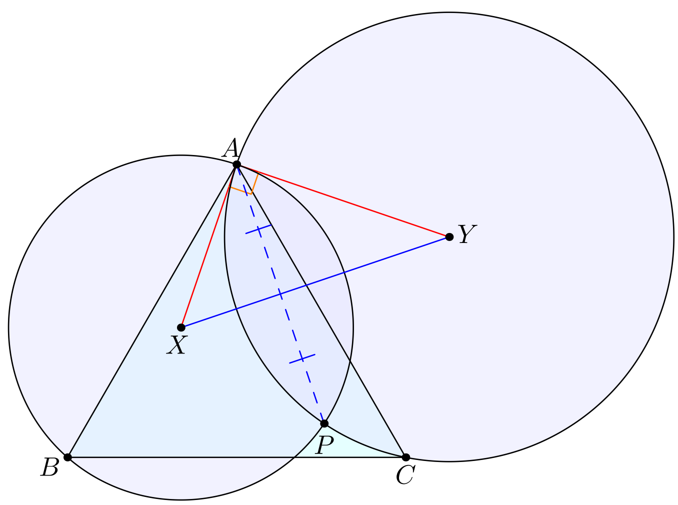
$$a_1^4+a_2^4+a_3^4+a_4^4+a_5^4=100-25\left(\frac{-20}{-25}\right)=\boxed{80}.$$

17. **[10]** Let P be a point inside equilateral triangle ABC such that $\angle BPC=150^\circ$. Given that circumradii of triangle ABP and triangle ACP are 3 and 5, respectively, compute AP.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{30}{\sqrt{34}}=\frac{15\sqrt{34}}{17}}$

Solution:



Let X and Y be the centers of $\odot(ABP)$ and $\odot(ACP)$. Write

$$\begin{aligned}
\angle XAY &= \angle XAP+\angle PAY\\
&= (90^\circ-\angle ABP)+(90^\circ-\angle ACP)\\
&= (\angle PBC+30^\circ)+(\angle PCB+30^\circ)\\
&= 60^\circ+(180^\circ-\angle BPC)=90^\circ.
\end{aligned}$$

Thus, $XY = \sqrt{3^2 + 5^2} = \sqrt{34}$. Note that AP is twice the height from A onto XY by symmetry, so

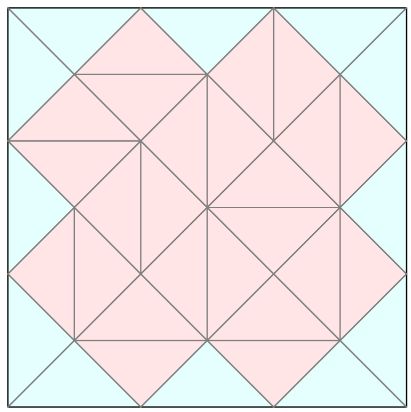
$$AP = \frac{4[AXY]}{XY} = \boxed{\frac{15\sqrt{34}}{17}}.$$

18. **[10]** Compute the number of ways to divide a 6×6 square into 36 triangles, each of which has side lengths $\sqrt{2}$, $\sqrt{2}$, and 2. (Rotations and reflections of a division are considered distinct divisions.)

Proposed by: Derek Liu

Answer: $\boxed{4096}$

Solution:



The square's edges must be lined with 12 triangles' hypotenuses because sides of $\sqrt{2}$ cannot add to an integer length. After placing those 12 triangles, rotate the diagram 45° and observe that the remaining area can be divided into 12 squares, each of side length $\sqrt{2}$. Each triangle must cover half of one of these squares by the lemma below (scaled up by a factor of $\sqrt{2}$). Each square can be covered in 2 ways, so there are $\boxed{2^{12}}$ ways to cover the entire figure.

Lemma 1. Consider any finite polyomino in a unit-length grid. If the polygon is divided into triangles of side lengths 1, 1, and $\sqrt{2}$, then each triangle covers half of one cell of the grid.

Proof. Align the grid so that the gridlines are horizontal and vertical. We use the term "edge" to refer to any edge of any triangle. Clearly, every edge is either horizontal, vertical, or diagonal (45° angle to horizontal).

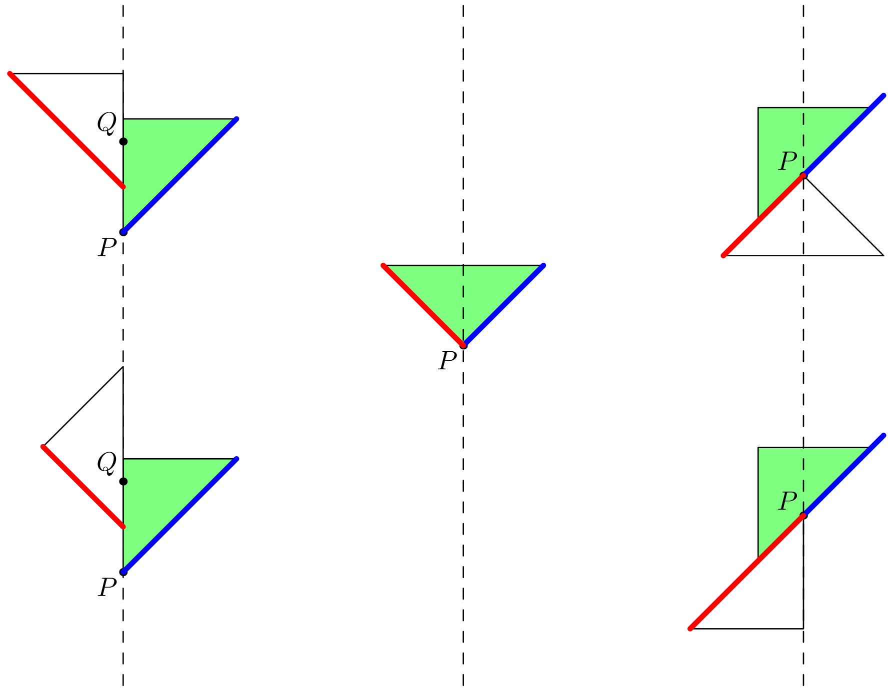
We first prove the following: for any vertical line ℓ which is not a grid line, if some diagonal edge e_1 has left endpoint on ℓ, then there also exists a diagonal edge with right endpoint on ℓ.

Without loss of generality, assume e_1 is in the up-right direction, and let P be the left endpoint of e_1. Consider the triangle $\mathcal{T}$ that has P on its perimeter and borders (some part of) edge e_1 from above. We consider three cases for $\mathcal{T}$.

- $\mathcal{T}$ has a 45° angle at P. Thus, it has a vertical edge at P. Let Q be any point on the interior of this edge. Since ℓ is not a gridline, some triangle on the left of ℓ has Q on its perimeter; this triangle has at least one diagonal edge with right endpoint on ℓ.
- $\mathcal{T}$ has a 90° angle at P. Then, $\mathcal{T}$ has an up-left edge from P, which is a diagonal edge with right endpoint on ℓ.
- $\mathcal{T}$ has a 180° angle at P (i.e., P is not a vertex of $\mathcal{T}$). Since P is a vertex of some triangle, there must exist a triangle with P as a vertex and a down-left edge from P (bordering $\mathcal{T}$ from below); this edge is a diagonal edge with right endpoint on ℓ.

In every case, the desired diagonal edge exists.

Some examples of each case are shown below (each column corresponds to one case). In each, ℓ is dashed, e_1 is drawn in blue, $\mathcal{T}$ is shaded green, and the desired edge is drawn in red.



Now assume for sake of contradiction that there exists a diagonal edge e_0 of length 1. Let ℓ_0 be the vertical line containing the left endpoint of e_0. If ℓ_0 is not a gridline, then there exists a diagonal edge e_1 with right endpoint on ℓ_0. Let ℓ_1 be the vertical line containing the left endpoint of e_1; if it is not a gridline, there exists a diagonal edge e_2 with right endpoint on ℓ_1. We continue this process; it cannot go on forever because the polyomino is finite, so eventually we reach some edge e_n with left endpoint on a vertical gridline.

We similarly extend rightwards, giving us a sequence of edges $e_n, e_{n-1}, \ldots, e_0, \ldots, e_{-m}$ such that the right endpoint of e_i and the left endpoint of e_{i-1} lie on the same vertical line, and the left endpoint of e_n and the right endpoint of e_{-m} both lie on vertical gridlines. Let d be the distance between these two gridlines; the total length of the edges in the sequence must be $d\sqrt{2}$. Every edge has length 1 or $\sqrt{2}$, so the total length can also be written in the form $a + b\sqrt{2}$; since e_0 has length 1, we know $a > 0$. Then $a = (d - b)\sqrt{2}$, contradicting the irrationality of $\sqrt{2}$.

Thus, every edge of length 1 is horizontal or vertical, so every edge of length $\sqrt{2}$ is diagonal. Hence, for any diagonal edge, the vertical lines through its two endpoints are separated by a distance of 1. If any diagonal edge had endpoints not on vertical gridlines, then constructing the same sequence of edges as before would not terminate, as for all i, ℓ_i would be exactly one unit left of ℓ_{i-1}. Thus, every diagonal edge has both endpoints on vertical gridlines.

A similar argument shows both endpoints are on horizontal gridlines as well. Thus, for any triangle in the division, its edge of length $\sqrt{2}$ is the diagonal of some cell of the grid. Thus, the triangle is half of this cell. $\square$

19. **[11]** Compute the number of ordered triples of positive integers (a, b, c) such that b is a divisor of 2025 and $\frac{a}{b} + \frac{b}{c} = \frac{a}{c}$.

Proposed by: Derek Liu, Pitchayut Saengrungkongka

Answer: $\boxed{105}$

Solution: Multiplying both sides by bc, we get $ac + b^2 = ab$, so $b^2 = a(b-c)$. Note that it is necessary for $d = b - c$ to be a divisor of b^2 strictly less than b. Conversely, for any b and $d < b$ dividing b^2, there exists exactly one such solution (a, b, c), namely, $(b^2/d, b, b-d)$. It remains to sum up the number of divisors of b^2 less than b for all divisors b of 2025.

The divisors of b^2 other than b can be paired up with each pair multiplying to b^2. Each pair contains one divisor greater than b and one divisor less than b, so if b^2 has s divisors, exactly $(s-1)/2$ of them are strictly less than b. Any divisor $b \mid 2025$ is of the form $3^x 5^y$ for nonnegative integers $x \le 4$ and $y \le 2$, in which case b^2 has $(2x+1)(2y+1)$ divisors. Thus, the answer is

$$\sum_{x=0}^{4}\sum_{y=0}^{2}\frac{(2x+1)(2y+1)-1}{2} = \frac{\left(\sum_{x=0}^{4}\sum_{y=0}^{2}(2x+1)(2y+1)\right)-15}{2} = \frac{(1+3+5+7+9)(1+3+5)-15}{2} = \boxed{105}.$$

Remark. If 2025 is replaced by n, the answer is $\frac{\tau(n)^2-\tau(n)}{2}$, where $\tau(n)$ is the number of divisors of n. This follows from the identity

$$\sum_{d\mid n}\tau(d^2) = \tau(n)^2,$$

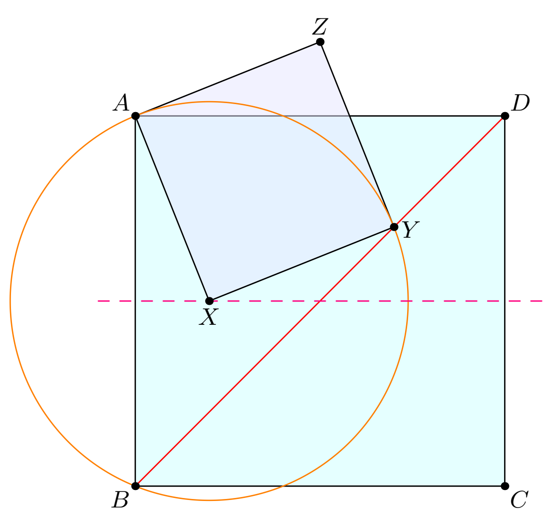
which can be easily shown by noting that both sides are multiplicative, so it suffices to check this at prime powers, which is easy.

20. **[11]** Suppose that $ABCD$ and $AXYZ$ are squares with side lengths 10 and 7, respectively. Given that X lies inside triangle ABY and Y lies on segment $\overline{BD}$, compute the area of triangle BXC.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{25}$

Solution:



Since $\angle ABY = 45^\circ$ and $\angle AXY = 90^\circ$, we get that X is the circumcenter of triangle ABY. Therefore, $XA = XB$, so X lies on the midsegment between AD and BC. Hence, the answer is $\frac{1}{2} \cdot 10 \cdot 5 = \boxed{25}$.

21. **[11]** Sarunyu starts at a vertex of a regular 7-gon. At each step, he chooses an unvisited vertex uniformly at random and walks to it along a straight line. He continues until all vertices are visited, and then walks back to his starting vertex along a straight line. A self-intersection occurs when two of his steps cross strictly inside the 7-gon. Compute the expected number of self-intersections in Sarunyu's walk.

Proposed by: Sebastian Attlan

Answer: $\boxed{\frac{14}{3}}$

Solution: Any intersection of two diagonals can be defined by the 4 points that they correspond to. Label the vertices of the 7-gon as A_1 through A_7 in cyclic order. For each group of 4 points A_i, A_j, A_k, A_l, with $1 \le i < j < k < l \le 7$, we compute the probability they are traversed with a self-intersection.

A self-intersection occurs when Sarunyu's walk visits these four vertices out of cyclic order. In other words, his path goes through A_i and A_k consecutively (in either order) and through A_j and A_l consecutively (in either order). Fix these 4 vertices and assume (without loss of generality) that Sarunyu's starting point is not among them. There are $6! = 720$ possible paths for the remaining 6 vertices in the walk. To count the paths that contain this intersection, we treat A_iA_k and A_jA_l as blocks for $4!$ ways to arrange the vertices and 2^2 ways to orient the blocks. This gives us $4! \cdot 2^2 = 96$ paths. Thus, the probability any set of 4 vertices gives an intersection is $\frac{96}{720} = \frac{2}{15}$. Applying linearity of expectation and summing over all choices of (i, j, k, l), the total expectation is

$$\frac{2}{15} \cdot \binom{7}{4} = \boxed{\frac{14}{3}}.$$

22. **[12]** Suppose that a, b, and c are pairwise distinct nonzero complex numbers such that

$$a^3 - 4a^2 + 5bc = b^3 - 4b^2 + 5ac = c^3 - 4c^2 + 5ab = 67.$$

Compute abc.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{42}$

Solution 1: For convenience, let $p = abc$. Then we have

$$x^4 - 4x^3 - 67x + 5p = 0 \quad \text{for } x \in \{a, b, c\}.$$

Thus, this quartic equation has a, b, c as roots, but the product of roots is $5p$, so 5 must be the last root. This implies that

$$5^4 - 4 \cdot 5^3 - 67 \cdot 5 + 5p = 0 \implies p = 67 + 4 \cdot 5^2 - 5^3 = \boxed{42}.$$

Solution 2: Taking the difference of the first two equations,

$$\begin{aligned} 0 = (a^3 - 4a^2 + 5bc) - (b^3 - 4b^2 + 5ac) &= (a^3 - b^3) - 4(a^2 - b^2) + 5c(b - a) \\ &= (a - b)(a^2 + ab + b^2) - 4(a - b)(a + b) - 5c(a - b) \\ &= (a - b)[(a^2 + ab + b^2) - 4(a + b) - 5c] \\ &= (a^2 + ab + b^2) - 4(a + b) - 5c, \end{aligned}$$

and similar for b, c and a, c, since a, b, c are not equal. Then, take the difference of these equations:

$$\begin{aligned}
0 &= [(a^2+ab+b^2)-4(a+b)-5c]-[(a^2+ac+c^2)-4(a+c)-5b] \\
&= (b^2-c^2)+a(b-c)-4(b-c)-5(c-b) \\
&= (b-c)[(b+c)+a+1] \\
&= b+c+a+1,
\end{aligned}$$

so $a+b+c=-1$. Taking the cyclic sum of the first difference equation, we get

$$\begin{aligned}
0 &= [(a^2+ab+b^2)-4(a+b)-5c]+[(b^2+bc+c^2)-4(b+c)-5a]+[(a^2+ac+c^2)-4(a+c)-5b] \\
&= 2(a^2+b^2+c^2)+(ab+bc+ca)-13(a+b+c) \\
&= 2[(a+b+c)^2-2(ab+bc+ca)]+(ab+bc+ca)-13(a+b+c) \\
&= 2[1-2(ab+bc+ca)]+(ab+bc+ca)+13 \\
&= -3(ab+bc+ca)+15,
\end{aligned}$$

so $ab+bc+ca=5$. Finally, sum the original expressions:

$$\begin{aligned}
67\cdot 3 &= (a^3-4a^2+5bc)+(b^3-4b^2+5ac)+(c^3-4c^2+5ab) \\
201 &= a^3+b^3+c^3-4(a^2+b^2+c^2)+5(ab+bc+ac) \\
&= [(a+b+c)^3-3(a^2b+a^2c+b^2a+b^2c+c^2b+c^2a)-6abc] \\
&\quad -4((a+b+c)^2-2(ab+bc+ca))+5\cdot 5 \\
&= [-1-3[(a+b+c)(ab+bc+ca)-3abc]-6abc]-4(1-2\cdot 5)+25 \\
&= [-1-3[-5-3abc]-6abc]+36+25 \\
&= -1+15+3abc+61 \\
&= 3abc+75.
\end{aligned}$$

Therefore, $abc = \boxed{42}$ from the equation $201 = 3abc+75$ above.

23. **[12]** Jacopo and Srinivas are playing a game with a bag of marbles. The bag starts with 6 red marbles and 6 blue marbles. Jacopo begins by drawing a marble from the bag, uniformly at random. When either player draws a marble, if it is red, the same player draws the next marble; otherwise, the other player draws the next marble (uniformly at random). All marbles are drawn without replacement. This process continues until all 12 marbles have been drawn. Compute the expected number of marbles that Jacopo draws.

Proposed by: Derek Liu

Answer: $\boxed{\frac{45}{7}}$

Solution: The players must take turns drawing the blue marbles, so Jacopo will always draw 3 blue marbles. Any given red marble will be drawn after either 0, 1, ..., or 6 blue marbles, all with equal probability; if this quantity is even, Jacopo draws it, and otherwise Srinivas draws it. Hence, there is a $\frac{4}{7}$ probability Jacopo draws any given red marble, so by linearity of expectation, as there are 6 red marbles, Jacopo draws an expected

$$6\cdot\frac{4}{7}=\frac{24}{7}$$

red marbles. By linearity of expectation, the total expected number of marbles Jacopo draws is equal to the sum of the expected numbers of blue and red marbles he draws, for an answer of

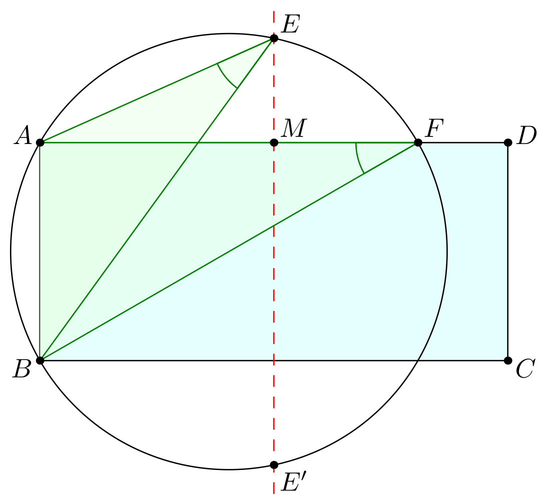
$$3+\frac{24}{7}=\boxed{\frac{45}{7}}.$$

24. **[12]** Let $ABCDE$ be a convex pentagon such that $ABCD$ is a rectangle and $\angle AEB = \angle CED = 30^\circ$. Given that $AB = 14$ and $BC = 20\sqrt{3}$, compute the area of triangle ADE.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{60\sqrt{3}}$

Solution 1:

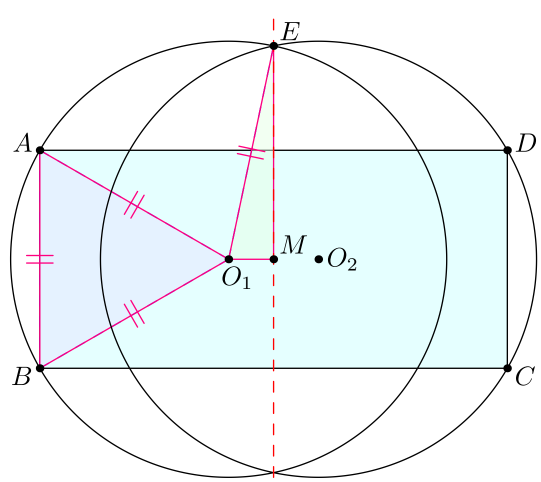


Let M be the midpoint of AD. Let $\odot(ABE)$ intersect AD and EM again at F and E', respectively.

Notice EM is the perpendicular bisector of AD, as $\angle AEB = \angle CED$. Then, $\angle AFB = \angle AEB = 30^\circ$, so $AF = AB \cdot \sqrt{3} = 14\sqrt{3}$. Since M is the midpoint of AD, we have $AM = 10\sqrt{3}$ and $MF = 4\sqrt{3}$.

By Power of a Point, $E'M \cdot ME = AM \cdot MF = 4\sqrt{3} \cdot 10\sqrt{3} = 120$. Furthermore, $E'E \parallel AB$ and $ABE'E$ is cyclic, so it is an isosceles trapezoid. Hence, $E'M = AB + EM = 14 + EM$ and $EM \cdot (EM + 14) = 120$, so $EM = 6$. The area of $\triangle ADE$ is then $1/2 \cdot 6 \cdot 20\sqrt{3} = \boxed{60\sqrt{3}}$.

Solution 2:



Let O_1 be the circumcenter of $\triangle EAB$, O_2 be the circumcenter of $\triangle EDC$, and M be the center of $ABCD$. We have that $\angle AO_1B = 2\angle AEB = 60^\circ$ and $O_1A = O_1B$, so $\triangle AO_1B$ is equilateral.

Since ME is the perpendicular bisector of AD and O_1M is the perpendicular bisector of AB, $\triangle O_1ME$ is a right triangle with $O_1E = O_1B = AB = 14$ and

$$O_1M = \operatorname{dist}(M, AB) - \operatorname{dist}(O_1, AB) = \frac{20\sqrt{3}}{2} - \frac{14\sqrt{3}}{2} = 3\sqrt{3}.$$

Thus, $EM = \sqrt{14^2 - \left(3\sqrt{3}\right)^2} = 13$, so the distance from E to AD is $13 - \frac{14}{2} = 6$, and the area of $\triangle ADE$ is $\frac{1}{2} \cdot 6 \cdot 20\sqrt{3} = \boxed{60\sqrt{3}}$.

25. **[13]** Jacob finished a rather hectic 9 holes of golf: he scored a 1 on the first hole, a 2 on the second hole, and so on, ending with a 9 on the ninth hole. Each hole is assigned a par of 3, 4, or 5. For each hole, Jacob subtracts its par from his score on that hole. Given that these 9 (possibly negative) differences can be arranged to form a sequence of 9 consecutive integers, compute the number of ways the pars could have been assigned to the holes.

Proposed by: Sebastian Attlan

Answer: $\boxed{57}$

Solution: We do casework on the smallest difference, which is at least $1 - 5 = -4$ and at most $1 - 3 = -2$.

- If the smallest difference is -4, then the differences must be $-4, -3, -2, \ldots, 4$, forcing all the pars to be 5.
- If the smallest difference is -2, then the differences must be $-2, -1, 0, \ldots, 6$, forcing all the pars to be 3.
- If the smallest difference is -3, then the differences must be $-3, -2, -1, \ldots, 5$. Let $f(n)$ denote the number of ways to choose the pars for holes $1, 2, \ldots, n$ such that the differences are $-3, -2, -1, \ldots, n-4$. We have $f(1) = 1$ (we must set the par to 4) and $f(2) = 2$ (we can set the pars to $3, 5$ or $4, 4$). We wish to compute $f(9)$.

  We derive a recursion for $f(n)$. Assume $n \geq 3$.

  - The par of the nth hole cannot be 3 because that would result in a difference of $n - 3$.
  - If the par of the nth hole is 4, then we need to choose the pars for holes $1, 2, \ldots, n-1$ such that the differences are $-3, -2, -1, \ldots, n-5$. There are $f(n-1)$ ways to do so.
  - If the par of the nth hole is 5, then consider the $(n-1)$th hole. If it has a par of 4, then both the $(n-1)$th and nth differences would be $n-4$, which is impossible. If it has a par of 5, then one of the first $n-2$ differences must be $n-4$, which is also impossible. Therefore, the $(n-1)$th hole must have a par of 3, meaning the $(n-1)$th and nth differences are $n-4$ and $n-5$. We now need to choose the pars for the holes $1, 2, \ldots, n-2$ such that the differences are $-3, -2, \ldots, n-6$. There are $f(n-2)$ ways to do so.

  Therefore, $f(n)$ satisfies the Fibonacci recursion. We can compute

  $$f(3) = 3,\ f(4) = 5,\ f(5) = 8,\ f(6) = 13,\ f(7) = 21,\ f(8) = 34,\ f(9) = 55.$$

  Hence, there are 55 possibilities in this case.

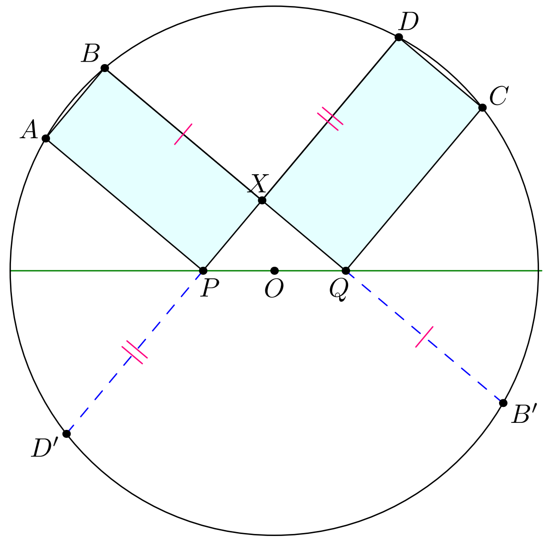
Finally, the answer is $1 + 1 + 55 = \boxed{57}$.

26. **[13]** Rectangles $ABXP$ and $CDXQ$ lie inside semicircle $\mathcal{S}$ such that A, B, C, and D lie on the arc of $\mathcal{S}$, and P and Q lie on the diameter of $\mathcal{S}$. Given that $BX = 7$, $PX = 6$, and $QX = 8$, compute DX.

Proposed by: Marin Hristov Hristov, Sarunyu Thongjarast

Answer: $\boxed{\sqrt{114} - 3}$

Solution:



Let O be the circumcenter of $\mathcal{S}$, which is the intersection of the perpendicular bisectors of AB and CD. The perpendicular bisector of AB is also the perpendicular bisector of PX, and the perpendicular bisector of CD is also the perpendicular bisector of QX. Therefore O is the intersection of the perpendicular bisectors of PX and QX. However, O lies on PQ; therefore $\angle PXQ = 90°$. Thus B, X, and Q are collinear, and similarly, D, X, and P are collinear.

Let Γ be the circle that contains $\mathcal{S}$. Let B' as the second intersection of BX and Γ, and D' be the second intersection of DX and Γ. Since O lies on the perpendicular bisector of QX and BB', we have that $BX = QB' = 7$, and similarly $DX = PD'$. Let $x = DX$. Then power of a point on X gives

$$7(8+7) = BX \cdot XB' = DX \cdot XD' = x(6+x).$$

Solving for x gives $x = \pm\sqrt{114} - 3$. The positive solution is $DX = \boxed{\sqrt{114} - 3}$.

27. **[13]** Let a_1, a_2, a_3, $\ldots$ be a sequence of integers such that $a_1 = 2$ and $a_{n+1} = a_n^7 - a_n + 1$ for all $n \geq 1$. Compute the remainder when a_{500} is divided by 7^3.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{274}$

Solution: Let $p = 7$. Notice that for all $n \geq 1$,

$$a_{n+1} \equiv a_n^p - a_n + 1 \equiv a_n - a_n + 1 \equiv 1 \pmod{p}.$$

Therefore, there exists a sequence of integers $x_2, x_3, \ldots$ such that we can write $a_{n+1} = 1 + px_{n+1}$ for all $n \geq 1$. Plugging this into the recurrence, we have

$$\begin{aligned}
1 + px_{n+1} &= a_{n+1} \\
&= a_n^7 - a_n + 1 \\
&= (1 + px_n)^p - (1 + px_n) + 1 \\
&= \left(\cdots + \binom{p}{2}(px_n)^2 + \binom{p}{1}(px_n) + \binom{p}{0}\right) - px_n \\
&\equiv p^2 x_n - px_n + 1 \pmod{p^3} \\
&\equiv 1 + p((p-1)x_n) \pmod{p^3}.
\end{aligned}$$

By repeatedly applying this fact, we get

$$1 + px_n \equiv 1 + p((p-1)^{n-2}x_2) \pmod{p^3}$$

for all $n \geq 2$.

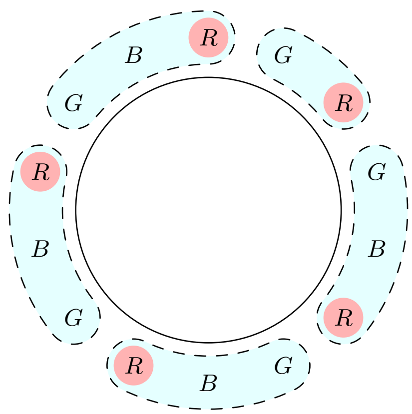
We have $a_2 = 127 = 1 + 7 \cdot 18$, so $x_2 = 18$. Now, using the above identity,

$$\begin{aligned}
a_{500} &= 1 + px_{500} \\
&\equiv 1 + p((p-1)^{498}x_2) \\
&\equiv 1 + p \cdot 18 \cdot (p-1)^{498} \\
&\equiv 1 + p \cdot 18 \cdot (1 - 498p + \cdots) \\
&\equiv 1 + p \cdot 18 \cdot (1 - 498p) \\
&\equiv 1 + 7 \cdot 18 - 7^2 \cdot 18 \cdot 498 \\
&\equiv 1 + 7 \cdot 18 - 7^2 \cdot 4 \cdot 1 \\
&\equiv 127 - 196 \equiv -69 \equiv \boxed{274} \pmod{p^3}.
\end{aligned}$$

28. **[15]** Let S be the set of integers between 0 and 100 (inclusive). Compute the number of ways to color each element of S either red, green, or blue such that for all elements x and y of S with $|x - y| - 1$ divisible by 3, the colors of x and y are different.

Proposed by: Derek Liu

Answer: $\boxed{606}$



Example of blocking when there are only 14 integers.

Solution 1: We work modulo 101, arranging the numbers in a circle. Since $101 \equiv 2 \bmod 3$, if the clockwise distance between two numbers is $1 \bmod 3$, so is the counterclockwise distance between them.

Some color must appear at least $\lceil 101/3 \rceil = 34$ times, dividing the circle into at least 34 intervals, which we will call *blocks* for clarity. No block can have length $1 \bmod 3$ by definition. Furthermore, at most one block can have length $2 \bmod 3$; otherwise, any interval spanning exactly two blocks of length $2 \bmod 3$ (so the rest are length $0 \bmod 3$) would have total length $1 \bmod 3$, which is disallowed. It is also clear that not every block can have length $0 \bmod 3$, as they have total length 101, so there must be exactly 1 block of length $2 \bmod 3$.

Since there are at least 34 blocks, their total length is at least $2 + 33 \cdot 3 = 101$. Thus, equality holds, and this color appears exactly 34 times. It follows that of the remaining $101 - 34 = 67$ numbers, another color appears 34 times. Without loss of generality, we assume red and green appear 34 times each, with the red numbers being every multiple of 3. (Any possible arrangement of 34 red numbers can be reached from this one by rotation.)

By the argument above, there exists x for which both $x-1$ and $x+1$ are both green. If x is not red, then one of $x \pm 1$ is red, so x must be red. If x is neither 0 nor 99, then both $x-2$ and $x+2$ are blue, but they differ by 4 (or 97), which is disallowed. Thus $x = 0$ or $x = 99$. Both lead to valid colorings (with the green numbers being 1, 4, 7, ..., 100 and 2, 5, ..., 95, 98, 100, respectively); one is a rotation of the other with red and green swapped.

Thus, there is one coloring up to rotation and permutation of colors, or $3! \cdot 101 = \boxed{606}$ colorings total.

Solution 2: Without loss of generality, assume 0 is red and 1 is green; we will multiply by 6 at the end. Then, 2 is either red or blue.

- If 2 is red, then 3 is either green or blue.
  - If 3 is green, then 4 must be blue and 5 must be red. Then, 6 cannot be blue, because that would make 0, 3, and 6 all different colors, leaving no choices for 7. Thus, 6 must be green. Then, 7 must be blue and 8 must be red. Again, 9 cannot be blue, because that would make 0, 3, and 9 all different colors, leaving no choices for 10. By repeating this argument, we get that the entire coloring is fixed with the following repeating pattern: $RG(RGB)(RGB)(RGB)\dots$.
  - If 3 is blue, then 4 must be green. Then, 5 cannot be blue, because that would force 6 to be green, making 0, 3, and 6 all different colors, leaving no choices for 7. Thus, 5 must be red. Then, 6 cannot be green, because that would make 0, 3, and 6 all different colors, leaving no choices for 7. Thus, 6 must be blue. By repeating these arguments, we get that the entire coloring is fixed with the following repeating pattern: $RG(RBG)(RBG)(RBG)\dots$.

  Therefore, there are 2 colorings in this case.
- If 2 is blue, then consider splitting the coloring into an initial pair followed by several triples:

  $$(RG)(B\_\_)(\_\_\_)\dots(\_\_\_).$$

  One valid coloring is setting all triples to be BRG. Otherwise, consider the first triple that deviates from BRG.
  - If it starts with a color other than B, then it must start with R because the previous pair/triple ends with G. From here, it is easy to check every remaining triple is RGB.
  - If it starts with B and its second color is not R, then its second color must be G. From here, it is easy to check that the triple must be BGB, and all remaining triples must be RGB.
  - If it starts with BR and its third color is not G, then its third color must be B. From here, it is easy to check that the triple must be BRB, and all remaining triples must be RGB.

  Therefore, the number of colorings equals the number of positions where the coloring could first deviate from $RG(RBG)(RBG)\dots$, plus 1 for the case where no deviation occurs. Hence, there are $98 + 1 = 99$ colorings for this case.

Our final answer is $6 \cdot (99 + 2) = \boxed{606}$.

29. **[15]** Compute the smallest positive integer multiple of 10001 with all of its digits distinct (when written in base 10).

Proposed by: Derek Liu

Answer: $\boxed{2650134987}$

Solution: Any multiple of 10001 with 8 or fewer digits can be written as the same 4-digit number repeated twice (possibly with leading 0s), so it is guaranteed to have repeated digits. Therefore, we consider multiples with > 8 digits.

Any multiple of 10001 with 9 or 10 digits can be written in the form $\underline{abc}$, where b and c are 4-digit numbers (with leading zeros allowed) and a is a 1 or 2-digit number such that $10001 \mid a - b + c$. Hence, $a + c = b$ or $a + c = b + 10001$.

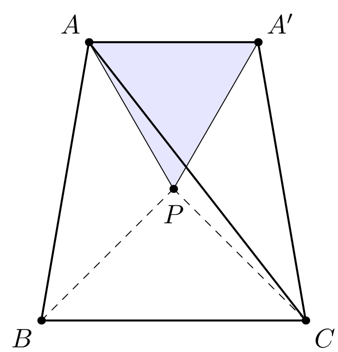
We cannot have $a+c=b+10001$, otherwise $c>9900$ and has repeated 9's. Thus, $a+c=b$. Suppose b starts with the digit d; then, c cannot start with d, but $c \geq b-99$, so c must start with the digit $e=d-1$. Furthermore, $b \geq \underline{d012}$ and $c \leq \underline{e987}$ because b and c have distinct digits, so $a \geq \underline{d012}-\underline{e987}=25$. If a starts with a 2, then $b \geq \underline{d013}$ to not repeat 2, but then a would have to be at least $\underline{d013}-\underline{e987}=26$. Thus, $a \geq 26$. There is exactly one solution with $a=26$, namely $26+4987=5013$, so the desired multiple is $\boxed{2650134987}$.

30. **[15]** Point P lies inside triangle ABC such that $BP=PC$ and $\angle APC - \angle APB = 60^\circ$. Given that $AP=12$, $AB=20$, and $AC=25$, compute BC.

Proposed by: Jason Mao

Answer: $\boxed{\frac{75}{4}=18.75}$

Solution: Let A' be the reflection of A about the perpendicular bisector of $\overline{BC}$.



Lemma 1. Quadrilateral $ABCA'$ is an isosceles trapezoid, and $\triangle APA'$ is equilateral.

Proof. Observe $\overline{A'C}$ is the reflection of $\overline{AB}$ about the perpendicular bisector of $\overline{BC}$, so $ABCA'$ must be an isosceles trapezoid. In particular, P lies on this perpendicular bisector, so

$$PA = PA' \quad \text{and} \quad \angle APA' = \angle APC - \angle A'PC = \angle APC - \angle APB = 60^\circ.$$

Thus $\triangle APA'$ is equilateral. □

Since $\triangle APA'$ is equilateral, we have $AA'=12$. Then Ptolemy's Theorem on $ABCA'$ yields

$$AA' \cdot BC + AB \cdot A'C = AC \cdot A'B \implies 12 \cdot BC + 20^2 = 25^2.$$

So $BC = \frac{25^2-20^2}{12} = \boxed{\frac{75}{4}}$.

31. **[17]** Gumdrops come in 7 different colors. Mark has two boxes of gumdrops, each containing one gumdrop of each color. He repeats the following process 7 times: he removes one gumdrop uniformly at random from each box, then eats one of the two removed gumdrops uniformly at random and throws away the other. Compute the probability Mark eats one gumdrop of each color.

Proposed by: Jason Mao

Answer: $\boxed{\frac{1}{16}}$

Solution 1: We consider the general problem where there are n different colored gumdrops. Without loss of generality, fix the order of gumdrops that Mark removes from the first box. There are $n!$ ways to arrange the order that Mark removes gumdrops from the second box with respect to the first, and 2^n ways for Mark to decide the gumdrops that he eats, for a total of $n! \cdot 2^n$ possible combinations.

To count the scenarios in which Mark eats all different colored gumdrops, assume Mark eats k gumdrops from the first box and $n-k$ gumdrops from the second box. There are $\binom{n}{k}$ ways to select which k colors of gumdrops from the first box are eaten. This uniquely determines the set of positions and colors of the $n-k$ gumdrops that Mark must eat from the second box. As for ordering the colors in the second box, there are $(n-k)!$ ways to permute the eaten gumdrops and $k!$ ways to permute the uneaten gumdrops. This is a total of $\binom{n}{k} \cdot (n-k)! \cdot k! = n!$ ways Mark can eat n different colored gumdrops for a given k.

Since k can be any integer from 0 to n, the total number of ways is $(n+1) \cdot n! = (n+1)!$, so the probability that Mark eats n different colored gumdrops is

$$\frac{(n+1)!}{n! \cdot 2^n} = \frac{n+1}{2^n}.$$

For $n = 7$, this gives an answer of $\boxed{\frac{1}{16}}$.

Solution 2: We consider the general problem where there are n different colored gumdrops. Label the gumdrop colors $1, 2, \ldots, n$, and suppose that gumdrop colored i from the first box is paired gumdrop colored $\sigma(i)$ in the second box.

We claim that condition on a fixed value of σ, the probability that Mark eats one gumdrop of each color is $2^{c(\sigma)-n}$, where $c(\sigma)$ is the number of cycles in σ. To see this, consider a cycle $i, \sigma(i), \sigma^2(i), \ldots, \sigma^{k-1}(i)$. Then in order for Mark to eat one gumdrop of each color, Mark has exactly two ways to eat the gumdrop of these colors:

- Eat all gumdrop of colors $i, \sigma(i), \sigma^2(i), \ldots$ from the first box.
- Eat all gumdrop of colors $i, \sigma(i), \sigma^2(i), \ldots$ from the second box.

Therefore, the number of ways for Mark to eat one gumdrop of each color is $2^{c(\sigma)}$. Thus, the claimed probability is $2^{c(\sigma)}$.

Therefore, the answer is

$$\frac{1}{2^n n!} \sum_{\sigma \in S_n} 2^{c(\sigma)},$$

where S_n is the set of all permutations of $\{1, 2, \ldots, n\}$. To evaluate the summation, we use the following well-known lemma, which is a property of Stirling numbers of first kind.

Lemma 1. For any n, we have

$$\sum_{\sigma \in S_n} x^{c(\sigma)} = x(x+1)(x+2)\ldots(x+n-1).$$

Proof. We use induction on n, with the base case $n = 1$ being clear. For the induction step, assume that the lemma is true for $n-1$. We split an element $\sigma \in S_n$ into two cases:

- If σ fixes n, then n is its own cycle, and we can remove n to get a permutation in S_{n-1}. Adding a lone cycle of n back multiply $x^{c(\sigma)}$ by x, so by induction hypothesis, the contribution of this case is
$$x \cdot x(x+1)(x+2)\ldots(x+n-2).$$
- If σ does not fix n (i.e., n is a part of a cycle), then we remove n from the cycle and reglue the cycle by changing $x \to n \to y$ to $x \to y$. This gives a permutation in S_{n-1}. For each permutation in S_{n-1}, there are $n-1$ ways to insert n back, so the contribution of this case is
$$(n-1) \cdot x(x+1)(x+2)\ldots(x+n-2).$$

Thus, the total sum is

$$(x+n-1) \cdot x(x+1)(x+2)\ldots(x+n-2),$$

completing the proof. $\square$

Plugging in $x = 2$ in the above lemma gives the answer

$$\frac{1}{2^n n!} \cdot (2 \cdot 3 \cdot 4 \cdots (n+1)) = \frac{n+1}{2^n},$$

and for $n = 7$, this evaluates to $\boxed{\frac{1}{16}}$.

32. **[17]** Compute the smallest positive integer n for which n^n (written in base 10) ends in 123.

Proposed by: Derek Liu

Answer: $\boxed{867}$

Solution: Note that $n^n \equiv 3 \bmod 8$, so $n \equiv 3 \bmod 8$. Thus,

$$n^3 \equiv n^n \equiv 3 \bmod 5,$$

so $n \equiv 27 \bmod 40$. Then,

$$n^7 \equiv n^n \equiv -2 \bmod 25.$$

Cubing both sides and using the fact that $n^{21} \equiv n \bmod 25$,

$$n \equiv (-2)^3 = -8 \bmod 25,$$

so $n \equiv 67 \bmod 200$. Then,

$$n^{67} \equiv n^n \equiv -2 \bmod 25.$$

Cubing both sides and using the fact that $n^{201} \equiv n \bmod 125$,

$$n \equiv (-2)^3 = -8 \bmod 125,$$

so $n \equiv \boxed{867} \bmod 1000$.

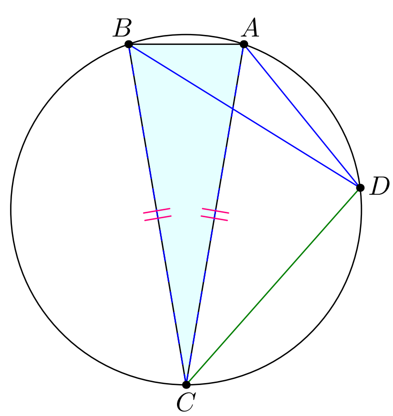
33. **[17]** Four points A, B, C, and D lie on a circle with radius 2 such that $CD = 3$, $CA = CB$, and $DA - DB = 1$. Compute the maximum possible value of AB.

Proposed by: Rohan Bodke

Answer: $\boxed{\frac{6\sqrt{3}}{7}}$

Solution: We have two cases on whether C is the midpoint of arc $\widehat{ADB}$ or not.

Case 1. If C is the midpoint of arc $\widehat{ADB}$, then $DA < DB$, and so $ABDC$ is a cyclic quadrilateral with vertices in that order.



Applying Ptolemy's on $ABCD$,

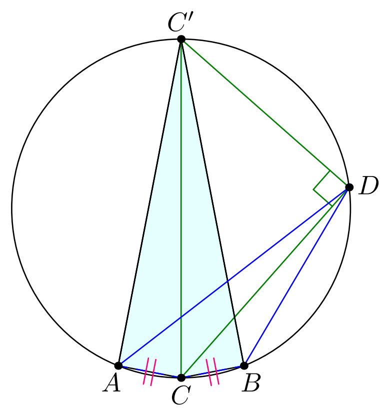
$$AB \cdot CD + AC \cdot BD = AD \cdot BC \implies 3AB = CA(DA - DB) = CA = CB.$$

Thus, $\triangle ABC$ is similar to one with side lengths 1, 3, and 3, and has circumradius 2. The area of a triangle with side lengths 1, 3, and 3 is $\frac{1}{2}\sqrt{3^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{35}}{4}$, so its circumradius is

$$\frac{1 \cdot 3 \cdot 3}{4 \cdot \frac{\sqrt{35}}{4}} = \frac{9}{\sqrt{35}}.$$

Thus, in order for $\triangle ABC$ to have a circumradius of 2, we must have $AB = \frac{2\sqrt{35}}{9}$.

Case 2. If C is not the midpoint of arc $\widehat{ADB}$, the direct application of Ptolemy's theorem on $ABCD$ is not useful.



The idea is to construct the antipode C' of C. We have

$$C'D = \sqrt{(CC')^2 - CD^2} = \sqrt{4^2 - 3^2} = \sqrt{7},$$

so we can apply a similar argument as before. Since $DA < DB$, we have that $AC'DB$ is cyclic in this order. By Ptolemy's theorem on $AC'DB$, we obtain

$$AC' \cdot DB + AB \cdot C'D = AD \cdot BC' \implies \sqrt{7}AB = BC'(DA - DB) = BC' = AC'.$$

Thus, $\triangle ABC'$ is similar to one with side lengths 1, $\sqrt{7}$, and $\sqrt{7}$ and has circumradius 2. The area of a 1-$\sqrt{7}$-$\sqrt{7}$ triangle is $\frac{1}{2}\sqrt{7 - \left(\frac{1}{2}\right)^2} = \frac{3\sqrt{3}}{4}$, so its circumradius is

$$\frac{1 \cdot \sqrt{7} \cdot \sqrt{7}}{4 \cdot \frac{3\sqrt{3}}{4}} = \frac{7\sqrt{3}}{9}.$$

Thus, in order for $\triangle ABC'$ to have a circumradius of 2, we must have $AB = \frac{6\sqrt{3}}{7}$.

Conclusion. The two possible values of AB are $\frac{6\sqrt{3}}{7}$ and $\frac{2\sqrt{35}}{9}$. However, we note that

$$\frac{6\sqrt{3}}{7} = \sqrt{\frac{108}{49}} > \sqrt{2} > \sqrt{\frac{140}{81}} = \frac{2\sqrt{35}}{9},$$

so $\boxed{\frac{6\sqrt{3}}{7}}$ is the maximum value.

34. **[20]** Estimate the number of integers in $\{1, 2, 3, \ldots, 10^8\}$ that can be written in the form $x^2 - 2025y^2$ for some integers x and y.

Submit a positive integer E. If the correct answer is A, you will receive $\left\lfloor 20.99 \max\left(0, 1 - \frac{|E-A|}{A}\right)^{2.5} \right\rfloor$ points.

Proposed by: Derek Liu, Pitchayut Saengrungkongka

Answer: $\boxed{6422804}$

Solution: Let $N = 10^8$, so $\ln N \approx 18$. Note that $x^2 - 2025y^2 = (x - 45y)(x + 45y)$ is a product of two numbers which are equivalent modulo 90. Conversely, if a and b are equivalent modulo 90, then $ab = x^2 - 2025y^2$ where $x = (a + b)/2$ and $y = (a - b)/90$. A naive estimate would be to simply count the number of such pairs (a, b) with $ab \leq N$. For each a, there are approximately $N/(90a)$ such b that work, so the total number of such pairs is around

$$\int_1^N \frac{N}{90a}\,\mathrm{d}a = \frac{1}{90} N \ln N \approx 2 \cdot 10^7.$$

Almost all pairs (a, b) pair up with (b, a) with the same product, so a first estimate might be half this quantity, or 10^7 (which earns 2 points).

However, this is an overestimate because it ignores the fact that two pairs (a, b) and (a', b') might consist of different values and still have the same product. A similar analysis as above shows that there are around $\frac{1}{4} N \ln N \approx 4.5 \cdot 10^8$ unordered pairs (a, b) of the same parity with product at most N, but there are only $3N/4$ possible differences of squares (any value which is not 2 mod 4). Thus, on average, each of these $3N/4$ differences is hit by around $4.5/(3/4) = 6$ pairs (a, b). Each pair has around a $1/45$ chance of satisfying $a \equiv b \bmod 90$ (as they are already equivalent mod 2), so the "probability" of any of these 6 pairs satisfying this is

$$1 - \left(1 - \frac{1}{45}\right)^6 \approx \frac{6}{45} - \frac{21}{2025} \approx 0.133 - 0.01 = 0.123.$$

This gives us a new estimate of

$$0.123 \cdot \left(\frac{3}{4} N\right) \approx 9.225 \cdot 10^6,$$

which earns 5 points.

This is still an overestimate because it assumes every number gets hit the same amount of times. However, any number of the form $x^2 - 2025y^2$ must be a square mod 2025. Note that there are $27 + 3 + 1 = 31$ squares mod 81 and $10 + 1 = 11$ mod 25, so the number of squares mod 2025 is $31 \cdot 11 = 341$, or around one-sixth of all numbers. These also make up for approximately $1/6$ of all such pairs (a, b), so each of these pairs has around a $6/45 = 2/9$ probability of satisfying $a \equiv b \bmod 90$. The new "probability" becomes

$$1 - \left(1 - \frac{2}{15}\right)^6 = 1 - \frac{13^6}{15^6} \approx 1 - \left(\frac{2200}{3300}\right)^2 = \frac{5}{9},$$

giving us an approximation of

$$\frac{5}{9}\left(\frac{1}{6} \cdot \frac{3}{4} N\right) \approx 6.94 \cdot 10^6,$$

which is a lot closer to the true value (earning 17 points).

This estimate still assumes that each of the 341 squares mod 2025 gets hit equally, but this isn't true either. For instance, 0 has 9 square roots mod 81 (the multiples of 9), but 1 only has 2 square roots. We can get a better estimate by applying the above method, but doing casework on $\gcd(ab, 2025)$ instead (note $a \equiv b \bmod 45$ implies this gcd is a square).

- Suppose $\gcd(ab, 2025) = 1$. We know $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$ of such pairs (a, b) satisfy this, and ab can be one of $27 \cdot 10 = 270$ residues mod 2025, so collisions are about $\frac{8}{15} \cdot \frac{2025}{270} = 4$ times more likely. The number of values we get this way will be around

$$\left(1 - \left(1 - \frac{4}{45}\right)^6\right)\left(\frac{3}{4} \cdot \frac{270}{2025} N\right) \approx 0.43 \cdot \frac{1}{10} N \approx 4.3 \cdot 10^6,$$

approximating $1 - (1 - 4/45)^6 \approx 6(4/45) - 15(4/45)^2 + 20(4/45)^3 \approx 0.53 - 0.12 + 0.02 = 0.43$. (Similar binomial expansions from now on will be omitted.)

- Suppose $\gcd(ab, 2025) = 9$. We know $\frac{2}{9} \cdot \frac{4}{5} = \frac{8}{45}$ of such pairs (a, b) satisfy this, and ab can be one of $3 \cdot 10 = 30$ residues mod 2025, so collisions are about $\frac{8}{45} \cdot \frac{2025}{30} = 12$ times more likely. The number of values we get this way will be around

$$\left(1 - \left(1 - \frac{12}{45}\right)^6\right)\left(\frac{3}{4} \cdot \frac{30}{2025} N\right) \approx 0.84 \cdot \frac{1}{90} N \approx 0.93 \cdot 10^6.$$

- Suppose $\gcd(ab, 2025) = 25$. We know $\frac{2}{3} \cdot \frac{1}{5} = \frac{2}{15}$ of such pairs (a, b) satisfy this, and ab can be one of $27 \cdot 1 = 27$ residues mod 2025, so collisions are about $\frac{2}{15} \cdot \frac{2025}{27} = 10$ times more likely. The number of values we get this way will be around

$$\left(1 - \left(1 - \frac{10}{45}\right)^6\right)\left(\frac{3}{4} \cdot \frac{27}{2025} N\right) \approx 0.78 \cdot \frac{1}{100} N \approx 0.78 \cdot 10^6.$$

- Suppose $\gcd(ab, 2025) = 81$. We know $\frac{1}{9} \cdot \frac{4}{5} = \frac{4}{45}$ of such pairs (a, b) satisfy this, and ab can be one of $1 \cdot 10 = 10$ residues mod 2025, so collisions are about $\frac{4}{45} \cdot \frac{2025}{10} = 18$ times more likely. The number of values we get this way will be around

$$\left(1 - \left(1 - \frac{18}{45}\right)^6\right)\left(\frac{3}{4} \cdot \frac{10}{2025} N\right) \approx 0.95 \cdot \frac{1}{270} N \approx 0.35 \cdot 10^6.$$

- One can similarly check that in the remaining cases $\gcd(ab, 2025) = 225$ and 2025, which make up for 4 residues mod 2025, collisions are about 36 and 45 times more likely, respectively. Since $(1 - 36/45)^6 \approx 0$, we may as well assume the number of values is

$$\frac{3}{4} \cdot \frac{4}{2025} N \approx 0.14 \cdot 10^6.$$

Adding these values up, we get

$$4.3 \cdot 10^6 + 0.93 \cdot 10^6 + 0.78 \cdot 10^6 + 0.35 \cdot 10^6 + 0.14 \cdot 10^6 = 6.5 \cdot 10^6,$$

which is accurate enough for all 20 points.

Apart from simply making better estimates, another further refinement that can be made is the following. Note that a number around x has approximately $\sum_{i=1}^{x} 1/x \approx \ln x$ divisors on average, as it has a $1/i$ chance of being divisible by i. Thus, we may assume that the number of times a product $n = ab$ is hit is proportional to $\ln x$. If we assume $x \not\equiv 2 \bmod 4$ is hit $c \ln x$ times, then the number of pairs (a, b) would be

$$\frac{3}{4} \int_1^N c \ln x \, \mathrm{d}x = \frac{3}{4} c [x \ln x - x]_{x=1}^N \approx \frac{3}{4} c N \ln N.$$

We know this quantity is around $\frac{1}{4} N \ln N$, so $c \approx 1/3$, i.e., x is hit around $(1/3) \ln x$ times. We can use this to approximate the number of values in each case as an integral instead; for example, the $\gcd(ab, 2025) = 1$ case would yield

$$\frac{3}{4} \cdot \frac{270}{2025} \int_1^N \left(1 - \left(1 - \frac{4}{45}\right)^{(1/3)\ln x}\right) \mathrm{d}x.$$

Note that the integrand is simply $1-x^m$ where $m = (1/3)\ln(1-4/45)$, so this integral can be computed fairly easily. (It evaluates to around $4.17 \cdot 10^6$, which is somewhat of an underestimate, but this mainly stems from the fact that our estimate for c is too high since we ignored the $-N$ term when integrating).

35. **[20]** Derek has a 45×45 grid of cells. Initially, every cell in the grid is uncolored. Every second, he picks one of the remaining uncolored cells uniformly at random and colors it in. He stops once there exist two distinct colored cells that share an edge. Estimate the expected number of seconds before Derek stops.

Submit a real number E. If the correct answer is A, you will receive $\lceil \max(0, 20 - 7|E-A|^{3/4}) \rceil$ points.

Proposed by: Derek Liu

Answer: $\boxed{\approx 29.267}$

Solution: Let $n = 45$. As a heuristic, after m cells are shaded, each pair of adjacent cells has around a $(m/n^2)^2$ probability of being colored. There are approximately $2n^2$ such pairs, so the expected number of pairs that are colored is $2n^2(m/n^2)^2 = 2m^2/n^2$. Since we stop after one such pair exists, we might expect this value to be 1, so $m \approx m/\sqrt{2} \approx 32$, which attains 6 points.

For a much better estimate, let X be the number of seconds that elapse. Let $p = \frac{2n(n-1)}{\binom{n^2}{2}} = \frac{4}{n(n+1)} \approx 0.002$ be the probability that two random distinct cells are adjacent. Assuming that this probability is independent for each pair (which is reasonable as this probability is very small), the probability that none of the first k colored cells are pairwise adjacent is $(1-p)^{k(k-1)/2}$. This is the probability that $X > k$, so

$$\mathbb{E}[X] \approx \sum_{k=0}^{\infty} \Pr[X > k] = \sum_{k=0}^{\infty} (1-p)^{k(k-1)/2}.$$

We can approximate this sum with an integral:

$$\mathbb{E}[X] \approx \int_{-1/2}^{\infty} (1-p)^{x(x-1)/2}\, \mathrm{d}x = (1-p)^{-1/8} \int_{-1/2}^{\infty} (1-p)^{(x-1/2)^2/2}\, \mathrm{d}x.$$

Note that $(1-p)^{-1/8}$ is extremely close to 1, so we ignore that factor. Let $c = \ln(1-p) \approx -p$. By shifting, this integral is

$$\int_{-1}^{\infty} (1-p)^{x^2/2}\, \mathrm{d}x = \int_{-1}^{\infty} e^{cx^2/2}\, \mathrm{d}x.$$

Performing a u-substitution of $u = x\sqrt{-c}$, this integral becomes

$$\frac{1}{\sqrt{-c}} \int_{\sqrt{-c}}^{\infty} e^{-x^2/2}\, \mathrm{d}x.$$

We know that $\int_0^{\infty} e^{-x^2}\, \mathrm{d}x = \sqrt{\pi/2}$. Since $\sqrt{-c} \approx 0$,

$$\int_{\sqrt{-c}}^{\infty} e^{-x^2/2}\, \mathrm{d}x \approx \sqrt{-c}e^{-0^2/2} + \int_0^{\infty} e^{-x^2/2}\, \mathrm{d}x = \sqrt{-c} + \sqrt{\frac{\pi}{2}},$$

so our estimation becomes

$$\frac{1}{\sqrt{-c}}\left(\sqrt{-c} + \frac{\sqrt{\pi}}{2}\right) = 1 + \frac{1}{\sqrt{-c}} \cdot \sqrt{\frac{\pi}{2}}.$$

Observe that

$$\frac{1}{\sqrt{-c}} \approx \frac{1}{\sqrt{p}} = \sqrt{\frac{n(n+1)}{4}} \approx \frac{n+0.5}{2} = \frac{91}{4},$$

while $\sqrt{\pi/2} \approx 5/4$ (as $\pi \approx 25/8$), so our estimate is

$$1 + \frac{91}{4} \cdot \frac{5}{4} \approx 29.44,$$

which gets 19 points (possibly 18, depending on the numerical estimations we use).

It turns out that $\sqrt{\pi/2}$ is actually around 1.253, and our estimate should be around 29.5 as a result. The reason we have an overestimate is that our independence assumptions work better when we assume cells are being colored with replacement (i.e., the same cell may be selected several times); the effect this has on p is negligible. Thus, it remains to estimate the number of distinct cells that are selected when 29.5 cells with replacement are selected on average. We assume the probability of a cell being selected 3 or more times is negligible, so the number of distinct cells is simply 29.5 minus the number of pairs of selected cells which are the same. Each pair of selected cells has a 1/2025 probability of being the same, and there are $(29.5 \cdot 28.5)/2$ such pairs, so by linearity of expectation, our new estimate is

$$29.5 - \frac{(29.5 \cdot 28.5)/2}{2025} \approx 29.29,$$

close enough for all 20 points.

36. **[20]** For each of the following properties, compute the smallest positive perfect square n with that property. (Unless otherwise specified, assume all numbers are written in base 10.)

1. n contains at least 4 of the same nonzero digit.
2. n starts with the same 2-digit number repeated twice (i.e., $\underline{abab}$ for digits a and b).
3. $n > 2025$ contains 2025 as a contiguous substring.
4. n contains 2520 as a contiguous substring.
5. The sum of the digits of n is 25.
6. When n is written in base 2025, the sum of its digits is 2025. (Submit n in base 10.)
7. $n > 10000$, and the last four digits of n are also a perfect square (possibly with leading 0s).
8. n^2 has 2025 positive divisors.
9. n has two positive divisors that sum to 2025.

Submit an 9-tuple of integers corresponding to the answers above, or an X for any value you wish to leave blank. For instance, if you think the first and last answers are 2025, you should submit "2025, X, X, X, X, X, X, X, 2025". If you submit C values and they are **all** correct, then you will receive $\lfloor C^2/4 \rfloor$ points. Otherwise, you will receive 0 points.

Proposed by: Derek Liu, Qiao Zhang

Answer: $\boxed{44944, 40401, 42025, 252004, 4489, 212521, 15625, 1587600, 3600}$

Solution: Almost all of these require some amount of casework, but clever optimization allows for minimal computation.

1. Clearly no 4-digit square has the same digit repeated 4 times. If there exists such a 5-digit square, either its last 2 digits are the same, or its first 3 digits are the same. (In either case, this digit is nonzero.)

   We consider the former case first. Observe no square ends in 22, 33, 77, or 88 because no square ends in 2, 3, 7, or 8. Furthermore, no square ends in 11, 55, 66, or 99 because no square is 2 or 3 modulo 4. Thus, such a square would have to end in 44. No square n ends in 4444, as otherwise $n/4$ would be a square ending in 11. Thus, we need only check squares of the form $\underline{4ab44}$ where either a or b is 4. It is easy to check that $\boxed{44944}$ is the only such square.

   For the latter case, it suffices to check squares starting with 111, 222, 333, and 444. The only such square is 22201, which does not satisfy the required property.

2. No 4-digit square works as it would have to be divisible by 101, so it suffices to check 5-digit squares.

   Observe that if n ends in the digit d, then $n \equiv d \bmod 101$. Suppose $n = m^2$. If $d = x^2$ is a square, then $m^2 - x^2$ is divisible by 1010, so one of $m \pm x$ is divisible by 101 and hence 202, and one of $m \pm x$ is divisible by 5 and hence 10. Furthermore, $x \leq 3$, so the smallest possible solution is $m = 201$ and $x = 1$, yielding $201^2 = \boxed{40401}$.

   Otherwise, $d = 5$ or $d = 6$. We can rule out $d = 5$ as follows. Recall $\underline{x5}^2 = \underline{y25}$ where $y = x(x+1)$, so y would have to be $\underline{a2a}$ for some digit $a \neq 0$. No such digit works (and only $a \leq 3$ needs to be checked).

   Ruling out $d = 6$ is trickier. The square roots of 6 mod 101 are 39 and 62, so m mod 101 would have to be one of these. It suffices to consider m between 100 and 201, so only $m = 140$ and 163 need to be checked; in either case, m^2 does not end in 6.

3. No 5-digit square starts with 2025, so it suffices to consider 5-digit squares ending in 2025. Recall that $\underline{x5}^2 = \underline{y25}$ where $y = x(x+1)$; here, we want y to end in 20. It can be seen that y cannot be 120, 220, or 320, but $420 = 20 \cdot 21$ works, so $n = \boxed{42025}$.

4. No 4-digit square works. No square ends in 2520 as no square only ends in one 0, and it can be checked no 5-digit square starts with 2520. Thus, it suffices to consider 6-digit squares. We know

   $$(502)^2 = 500^2 + 2 \cdot 2 \cdot 500 + 2^2 = \boxed{252004}$$

   works, so we only need to check no 6-digit squares start with 12520 or 22520, which is true.

5. Clearly, n has at least 4 digits. Furthermore, $n \equiv 7 \bmod 9$, so $\sqrt{n}$ is either 4 or 5 mod 9. Since $\sqrt{n} \geq 32$, the possible values of $\sqrt{n}$ to check are 32, 40, 41, 49, 50, 58, 59, and 67 (the last of which works).

   We can narrow this search further. If n^2 has 4 digits that sum to 25, then the first and last digits must sum to at least 7. Thus, of the above values, clearly only 58 and 67 need to be checked. $58^2 = 3364$ fails, but $67^2 = \boxed{4489}$ works.

6. Observe $n \equiv 2025 \equiv 1 \bmod 2024$, so $n = m^2$ is 1 mod 8, 11, and 23. The first of these simply means m is odd. We know m must be a square root of 1 mod 253; there are four possible square roots, corresponding to the values which are ± 1 mod both 11 and 23. Conveniently, we already know 1 and 45 are two such roots, so the other two are $253 - 45 = 208$ and $253 - 1 = 252$. Since 1 and 45 are too small for m, and the next four values are all even, the smallest m can be is $253 + 208 = 461$, yielding $n = \boxed{212521}$. Its base-2025 expansion need not be checked; since $2025 < n < 2024^2$ and $n \equiv 1 \bmod 2024$, we know $n = 2024a + 2025$ for some $1 \leq a \leq 2024$. If $b = 2025 - a$, then $b = 2025a + b = \underline{ab}_{2025}$, and its digit sum is $a + b = 2025$.

7. Let us suppose $n = a^2$ is a 5-digit square starting with 1, which is the smallest it could be. Let $n - 10000 = b^2$. Then, $(a - b)(a + b) = 10000$, so we want to write 10000 as a product of two factors $a - b$ and $a + b$ with the smallest possible sum. We know these two factors have the same parity (ruling out $80 \cdot 125$), so the best we can do is $50 \cdot 200$, giving us $a = 125$ and $n = \boxed{15625}$.

8. We know n^2 is a fourth power, so its prime factorization can be written as $p_1^{e_1} p_2^{e_2} \ldots$ where each e_i is 1 mod 4. Since $\prod_i (e_i + 1) = 2025 = 3^4 5^2$, we must write 2025 as a product of 1 mod 4 factors. The best way to do so is $9 \cdot 9 \cdot 5 \cdot 5$, yielding $n = 2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2 = 1260^2 = \boxed{1587600}$. (This is a lot better than $25 \cdot 9 \cdot 9$, which yields $2^{12} \cdot 3^4 \cdot 5^4 = 14400^2$.)

9. Observe that $1800 + 225 = 2025$, so $n = 3600$ works. Now assume for sake of contradiction that $n < 3600$. Since 2025 is odd, the two divisors that sum to 2025 must be distinct. Note that $(n/3) + (n/4) = 7n/12$, which is less than 2025 unless $n = 59^2$, which clearly fails; thus, one of the divisors must be either n or $n/2$.

   If n is one of the divisors, let the other one be n/d for some positive integer d. Then, $2025 = n + n/d = n \cdot (d+1)/d$, so $(d+1)/d = 2025/n$ must be a quotient of two squares. But $d + 1$ and d cannot both be squares, contradiction.

If $n/2$ is one of the divisors, let the other one be n/d for some positive integer d. Then,

$$2025 = \frac{n}{2} + \frac{n}{d} = n\left(\frac{d/2+1}{d}\right) \implies \frac{d/2+1}{d} = \frac{2025}{n}.$$

Since n is even, it is divisible by 4, so $n/2$ is even. We cannot have both $n/2$ and n/d even as they sum to 2025, so n/d is odd; hence, d is divisible by 4. It follows that $d/2+1$ and d are relatively prime (as the former is odd), so they are both squares. The above equation shows we want d to be as small as possible to minimize n; while $d = 4$ fails, $d = 16$ works, which yields $n = \boxed{3600}$. This covers both cases. Hence, no smaller n exists.