HMMT February 2026

February 14, 2026

Algebra and Number Theory Round

1. A line intersects the graph of $y = x^2 + \frac{2}{x}$ at three distinct points. Given that the x-coordinates of two of the points are 6 and 7, respectively, compute the x-coordinate of the third point.

   Proposed by: Pitchayut Saengrungkongka

   Answer: $\boxed{-\frac{1}{21}}$

   Solution: Let the equation of the line be $y = ax + b$. Then, the x-coordinates of the intersection points satisfy
   $$ax + b = x^2 + \tfrac{2}{x},$$
   or $x^3 - ax^2 - bx + 2 = 0$. We know that two of the solutions to this equation are $x = 6$ and $x = 7$. Furthermore, the product of all three solutions is -2 by Vieta's formula. Hence, the third solution is $\frac{-2}{6 \cdot 7} = \boxed{-\frac{1}{21}}$.

2. Compute the second smallest positive integer n such that

   - n is divisible by 101, and
   - the decimal representation of n contains the number 2026 as a consecutive sequence of digits.

   Proposed by: Jacopo Rizzo

   Answer: $\boxed{420261}$

   Solution: We proceed by casework on the position of the substring 2026 in n.

   - If n ends in 2026, then we may write $n = 10000x + 2026$ for some positive integer x. Then, the condition $101 \mid n$ simplifies to $101 \mid x + 6$. Hence, $x \geq 95$, so the smallest possibility in this case is 952026.
   - If n ends in 2026 followed by a digit, then we may write $n = 100000x + 20260 + y$, where x is a positive integer and y is a digit. The condition $101 \mid n$ simplifies to $101 \mid 10x + 60 + y$. Hence, $10x + y \geq 41$, so the smallest possibility in this case is 420261.
   - If n ends in 2026 followed by two digits, then we may write $n = 1000000x + 202600 + y$, where x is a positive integer and y is a two-digit positive integer. The condition $101 \mid n$ simplifies to $101 \mid (-x - 6 + y)$. If $x = 0$, then the only possibility is $y = 6$, yielding the solution 202606. Otherwise, $x > 0$, yielding solutions $n \geq 10^6$.
   - If n ends in 2026 followed by three or more digits, then $n \geq 10^6$.

   Thus, the smallest two values of n are 202606 and $\boxed{420261}$.

3. Compute the sum of all positive integers n such that n has at least 6 positive integer divisors and the 6th largest divisor of n is 6.

   Proposed by: Srinivas Arun

   Answer: $\boxed{48}$

   Solution: Since 6 is a divisor of n, we know that 1, 2, and 3 are also divisors of n. We proceed by casework on which of 4 and 5 are divisors of n.

   - If neither 4 or 5 are divisors of n, then the four smallest divisors of n are 1, 2, 3, and 6, and the four largest divisors of n are $\frac{n}{6}$, $\frac{n}{3}$, $\frac{n}{2}$, and n. Moreover, since 6 is the 6th largest divisor, we know that n has exactly one divisor between 6 and $\frac{n}{6}$. This implies that n has an odd number of divisors and is thus a perfect square, which is impossible because n is divisible by 2 but not 4.

- If 4 is a divisor of n but 5 is not, then the five smallest divisors of n are 1, 2, 3, 4, and 6, and the five largest divisors of n are $\frac{n}{6}$, $\frac{n}{4}$, $\frac{n}{3}$, $\frac{n}{2}$, and n. Moreover, since 6 is the 6th largest divisor, we know that there are no divisors between 6 and $\frac{n}{6}$. Therefore, n has exactly 10 divisors total. Since n is divisible by 3 and 4, we must have $n = 2^4 \cdot 3 = 48$, which is a solution.
- If 5 is a divisor of n but 4 is not, then the five smallest divisors of n are 1, 2, 3, 5, and 6, and the five largest divisors of n are $\frac{n}{6}$, $\frac{n}{5}$, $\frac{n}{3}$, $\frac{n}{2}$, and n. Moreover, since 6 is the 6th largest divisor, we know that there are no divisors between 6 and $\frac{n}{6}$. Therefore, n has exactly 10 divisors total, which is impossible because n has at least three distinct prime factors (2, 3, and 5).
- If both 4 and 5 are divisors of n, then the six smallest divisors of n are 1, 2, 3, 4, 5, and 6, and the six largest divisors of n are $\frac{n}{6}$, $\frac{n}{5}$, $\frac{n}{4}$, $\frac{n}{3}$, $\frac{n}{2}$, and n. Therefore, we must have $n = 36$. However, we assumed that n is divisible by 5 in this case, which leads to a contradiction.

Finally, since the only answer is $n = 48$, the final answer is $\boxed{48}$.

4. Let a, b, and c be pairwise distinct complex numbers such that

$$\begin{aligned} a^2 + ab + b^2 &= 3(a+b), \\ a^2 + ac + c^2 &= 3(a+c), \\ b^2 + bc + c^2 &= 5(b+c) + 1. \end{aligned}$$

Compute a.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{7}{2} = 3.5}$

Solution: We multiply the first and second equation by $(a-b)$ and $(a-c)$ respectively. From this, we get the equations

$$\begin{aligned} a^3 - b^3 &= 3a^2 - 3b^2, \\ a^3 - c^3 &= 3a^2 - 3c^2. \end{aligned}$$

In particular, these rearrange to $a^3 - 3a^2 = b^3 - 3b^2$ and $a^3 - 3a^2 = c^3 - 3c^2$. Therefore, there exists a complex number k, such that

$$a^3 - 3a^2 = b^3 - 3b^2 = c^3 - 3c^2 = k.$$

Thus, a, b, and c are the three roots of the cubic $x^3 - 3x^2 - k = 0$. By Vieta's formulas, we have $a + b + c = 3$ and $ab + bc + ca = 0$. Hence,

$$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 9.$$

Therefore, we have

$$\begin{aligned} 18 &= 2(a^2 + b^2 + c^2) + (ab + bc + ca) \\ &= (a^2 + ab + b^2) + (b^2 + bc + c^2) + (c^2 + ca + a^2) \\ &= 3(a+b) + 3(b+c) + 5(b+c) + 1 \\ &= 6(a+b+c) + 2(b+c) + 1 \\ &= 18 + 2(b+c) + 1. \end{aligned}$$

Therefore, $b + c = -\frac{1}{2}$, whence $a = 3 - (b+c) = \boxed{\frac{7}{2}}$.

5. Compute the largest positive integer n such that

$$n \text{ divides } (\lfloor \sqrt{n} \rfloor)!^{n!} + 450.$$

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{1230}$

Solution: We will characterize all positive integer n such that n divides $(\lfloor \sqrt{n} \rfloor)!^{n!} + k$, where k is a fixed positive integer.

Claim 1. We have n divides $(\lfloor \sqrt{n} \rfloor)!^{n!} + k$ if and only if one of the following holds:

- $n \mid k$; or
- $n = dp$, where $d \mid k$ and p is a prime such that $p > d$ and $p \mid k + 1$.

Proof. Let n be such that $n \mid (\lfloor \sqrt{n} \rfloor)!^{n!} + k$ and consider any prime divisor q of n.

- If $q \mid k$, then we note that
$$\nu_q((\lfloor \sqrt{n} \rfloor)!^{n!}) \geq n! > \nu_q(n),$$
so we must have that $\nu_q(n) \leq \nu_q(k)$.
- If q does not divide k, then q does not divide $(\lfloor \sqrt{n} \rfloor)!$, so $q > \lfloor \sqrt{n} \rfloor$. In particular, there exists at most one such q.

In particular, we have two cases.

- If every prime dividing n also divides k, then from the first bullet point, $n \mid k$.
- If there exists a prime p such that $p \mid n$ but $p \nmid k$, then $p > \lfloor \sqrt{n} \rfloor$. Let $n = pd$. Then $d < p$. Moreover, from the first bullet point, $\nu_q(d) \leq \nu_q(k)$ for all prime q. Thus, $d \mid k$. Finally, by Fermat's little theorem, $(\lfloor \sqrt{n} \rfloor)^{p-1} \equiv 1 \pmod{p}$, so $(\lfloor \sqrt{n} \rfloor)^{n!} \equiv 1 \pmod{p}$, which implies that $p \mid k + 1$. $\square$

Finally, we extract the answer. We can easily check that $1230 = 30 \cdot 41$ works because $30 \mid 450$, $41 \mid 451$, and $30 < 41$. We now have to show that this is the largest possible solution.

- If $n \mid 450$, then $n \leq 450$.
- If $n = kp$ where $k \mid 450$, $p \mid 451$, and $k \leq p$, then from $p \mid 451 = 11 \cdot 41$, we have that p is either 11 or 41.
  - If $p = 11$, then $n \leq 121$.
  - If $p = 41$, then we have to list all divisors of 450 that are less than 41. The largest one is 30, so $n \leq 30 \cdot 41 = 1230$.

Remark. For concreteness, the complete list of all n are

- (divisors of 450) 1, 2, 3, 5, 6, 9, 10, 15, 18, 25, 30, 45, 50, 75, 90, 150, 225, 450,
- (multiples of 11) 11, 22, 33, 55, 66, 99, 110.
- (multiples of 41) 41, 82, 123, 205, 369, 410, 615, 738, 1025, 1230.

6. The numbers 1, 2, …, 2100 are written on a board. Every second, Mark takes two numbers on the board, a and b, erases them, and replaces them with $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$. Mark stops once any move he makes will not change the numbers written on the board. Compute the number of divisors of the 2026th smallest positive integer written on the board when he finishes.

   Proposed by: Rishabh Das

   Answer: $\boxed{3840}$

   Solution: Let $p < 2100$ be a prime number. Consider the largest power of p dividing a positive integer n in the sequence (we'll call this the ν_p value of n, denoted $\nu_p(n)$). For a fixed p, performing an operation on two numbers a and b preserves the ν_p values of the numbers. Indeed, $\nu_p(\gcd(a, b)) = \min(\nu_p(a), \nu_p(b))$ and $\nu_p(\operatorname{lcm}(a, b)) = \max(\nu_p(a), \nu_p(b))$, so we always have

$$\{\nu_p(a), \nu_p(b)\} = \{\nu_p(\gcd(a, b)), \nu_p(\operatorname{lcm}(a, b))\}.$$

   Therefore, the multiset of ν_p values for the entire sequence never changes. Moreover, when Mark finishes, if the numbers on the board are $a_1 \leq a_2 \leq \cdots \leq a_{2100}$, then we must have $a_i \mid a_{i+1}$ for all $1 \leq i < 2100$, or else we would be able to perform an operation on a_i and a_{i+1}. Thus, when Mark finishes, we have

$$\nu_p(a_1) \leq \nu_p(a_2) \leq \cdots \leq \nu_p(a_{2100})$$

   for all primes p. Combining this with the fact that the multiset of ν_p values doesn't change under operations, we can conclude that the sequence $\nu_p(a_1)$, $\nu_p(a_2)$, …, $\nu_p(a_{2100})$ is the multiset of ν_p in the beginning arranged in nondecreasing order.

   Now, we want to compute a_{2026}, or the 75th largest number when Mark finishes. Since the ν_p values are sorted, we have that p^k divides a_{2026} if and only if there exist at least 75 numbers in the sequence divisible by p^k. This is equivalent to $2100/p^k \geq 75$, or $p^k \leq 28$. Hence, the prime powers that divide a_{2026} are $2^4, 3^3, 5^2, 7, 11, 13, 17, 19, 23$. Thus

$$a_{2026} = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23,$$

   which has $(4 + 1)(3 + 1)(2 + 1)(1 + 1)^6 = \boxed{3840}$ divisors.

7. Positive real numbers x, y, and z satisfy the following equations:

$$\begin{aligned} xyz &= 3, \\ (x-y)(y-z)(z-x) &= 4, \\ (x+y)(y+z)(z+x) &= 40. \end{aligned}$$

   Compute the minimum possible value for x.

   Proposed by: Derek Liu

   Answer: $\boxed{(3 + \sqrt{6})^{-1/3}}$

   Solution 1: Expanding the third equation gives

$$(x^2y + y^2z + z^2x) + (xy^2 + yz^2 + zx^2) + 2xyz = 40.$$

   Since $xyz = 3$, we have

$$(xy^2 + yz^2 + zx^2) + (x^2y + y^2z + z^2x) = 34.$$

   Expanding the second equation gives

$$(x^2z + y^2x + z^2y) - (x^2y + y^2z + z^2x) = 4.$$

The two equations above yield

$$x^2z + y^2x + z^2y = 19,$$
$$x^2y + y^2z + z^2x = 15.$$

Dividing both equations by $xyz = 3$, we conclude that

$$\tfrac{x}{y} + \tfrac{y}{z} + \tfrac{z}{x} = \tfrac{19}{3},$$
$$\tfrac{x}{z} + \tfrac{y}{x} + \tfrac{z}{y} = 5.$$

Note that $xyz = 3$, so x, y, and z are nonzero, and let $a = \frac{y}{x}$, $b = \frac{z}{y}$, and $c = \frac{x}{z}$. Then, $a + b + c = 5$ and $abc = 1$. Furthermore,

$$ab + bc + ca = \frac{1}{c} + \frac{1}{a} + \frac{1}{b} = \frac{19}{3}.$$

Therefore, a, b, and c are the roots of

$$t^3 - 5t^2 + \frac{19}{3}t - 1 = 0.$$

This cubic factors as

$$(t-3)\left(t^2 - 2t + \tfrac{1}{3}\right) = 0,$$

with roots 3, $1 + \sqrt{2/3}$, and $1 - \sqrt{2/3}$. Since $xyz = 3$ is fixed, for x to be minimized, both $\frac{y}{x}$ and $\frac{z}{x}$ should be maximized. Their maximal possible values are 3 and $1/(1 - \sqrt{2/3}) = 3 + \sqrt{6}$, respectively, so the minimum possible value for x satisfies

$$3\left(3 + \sqrt{6}\right)x^3 = 3 \implies x = \boxed{\left(3 + \sqrt{6}\right)^{-1/3}}.$$

Solution 2: Note that x, y, and z are nonzero because $xyz = 3$, and consider the cubic with roots $\frac{x}{y}$, $\frac{y}{z}$, and $\frac{z}{x}$. Let

$$f(t) = t^3 + ut^2 + vt + w = (t - \tfrac{x}{y})(t - \tfrac{y}{z})(t - \tfrac{z}{x}).$$

By Vieta's formula, $-w$ is the product of the roots of f, which is $(\frac{x}{y})(\frac{y}{z})(\frac{z}{x}) = 1$. Moreover, notice that

$$\begin{aligned} f(1) &= (1 - \tfrac{x}{y})(1 - \tfrac{y}{z})(1 - \tfrac{z}{x}) \\ &= (y-x)(z-y)(x-z)/(xyz) \\ &= -\frac{4}{3}, \end{aligned}$$

and

$$\begin{aligned} f(-1) &= (-1 - \tfrac{x}{y})(-1 - \tfrac{y}{z})(-1 - \tfrac{z}{x}) \\ &= (-x-y)(-y-z)(-z-x)/(xyz) \\ &= -\frac{40}{3}. \end{aligned}$$

Expressing these in terms of the coefficients of f yields the system

$$\begin{aligned} -w &= 1, \\ 1 + u + v + w &= -\frac{4}{3}, \\ -1 + u - v + w &= -\frac{40}{3}. \end{aligned}$$

Solving this system of equations gives

$$f(t) = t^3 - \frac{19}{3}t^2 + 5t - 1,$$

whence we can finish similar to the first solution.

8. Let $a_0, a_1, a_2, \ldots$ be the unique sequence of nonnegative integers less than 397 with $a_0 = 1$ and

$$a_{n+1}(a_n + 1)^2 \equiv a_n \pmod{397}$$

for all nonnegative integers n. Given that $a_{2026} = 9$, compute the remainder when $a_0 + a_1 + \cdots + a_{2026}$ is divided by 397.

Proposed by: Tiger Zhang

Answer: $\boxed{279}$

Solution: We work in modulo 397, and we will freely use division. (Formally, we will use the notation $\frac{a}{b}$ to refer to ab^{-1}, where b^{-1} is the multiplicative inverse of b modulo 397. Also, all equalities are in modulo 397.) First, we rewrite the given equation to

$$\begin{aligned} a_{n+1}(a_n + 1)^2 &= a_n \\ a_{n+1}a_n^2 + 2a_{n+1}a_n + a_{n+1} &= a_n \\ a_n + 2 + \frac{1}{a_n} &= \frac{1}{a_{n+1}} \\ a_n &= \frac{1}{a_{n+1}} - \frac{1}{a_n} - 2. \end{aligned}$$

In particular, we have that

$$\begin{aligned} a_0 &= \frac{1}{a_1} - \frac{1}{a_0} - 2 \\ a_1 &= \frac{1}{a_2} - \frac{1}{a_1} - 2 \\ &\vdots \\ a_{2026} &= \frac{1}{a_{2027}} - \frac{1}{a_{2026}} - 2, \end{aligned}$$

so adding them all gives

$$a_0 + a_1 + \cdots + a_{2026} = \frac{1}{a_{2027}} - \frac{1}{a_0} - 2 \cdot 2027.$$

Hence, since $a_{2027} = \frac{a_{2026}}{(a_{2026}+1)^2} = \frac{9}{100}$, we have

$$\begin{aligned} a_0 + a_1 + \cdots + a_{2026} &= \frac{100}{9} - 1 - 2 \cdot 2027 \\ &= 364 - 1 - 84 = \boxed{279}. \end{aligned}$$

9. Compute

$$\sum_{k=1}^{\infty} \left(2^{-\lfloor 101k/1 \rfloor} + 2^{-\lfloor 101k/2 \rfloor} + \cdots + 2^{-\lfloor 101k/100 \rfloor} \right).$$

Proposed by: Albert Wang, Isaac Zhu, Karthik Venkata Vedula

Answer: $\boxed{50\left(1-\frac{1}{2^{101}-1}\right)}$

Solution: For all positive integers n, let $f(n)$ be the number of pairs (k, j) of positive integers with $k \geq 1$ and $j \in \{1, 2, \ldots, 100\}$ such that $\left\lfloor \frac{101k}{j} \right\rfloor = n$. We show that:

- $f(n) = 100$ when $101 \mid n$,
- $f(n) = 0$ when $101 \mid n + 1$,
- $f(n) = 50$ otherwise.

The answer is then

$$\sum_{n\geq 1} \frac{f(n)}{2^n} = \sum_{n\geq 1} \frac{50}{2^n} - \sum_{n\geq 1} \frac{50}{2^{101n-1}} + \sum_{n\geq 1} \frac{50}{2^{101n}} = \boxed{50\left(1-\frac{1}{2^{101}-1}\right)}.$$

First, $\left\lfloor \frac{101k}{j} \right\rfloor = n$ is equivalent to

$$n+1 > \frac{101k}{j} \geq n \iff \frac{(n+1)j}{101} > k \geq \frac{nj}{101}.$$

For fixed j and n, note that there is at most one integer k satisfying the above inequality. We now take cases on n modulo 101.

- $101 \mid n$. Here, $\frac{nj}{101}$ is an integer and $\frac{(n+1)j}{101} < \frac{nj}{101} + 1$, thus for each $j \in \{1, 2, \ldots, 100\}$ there is exactly one integer k satisfying the above inequality. In this case, $f(n) = 100$.
- $101 \mid n + 1$. Since $\frac{nj}{101} < \frac{(n+1)j}{101} - 1$, it is clear no such k exists, so $f(n) = 0$.
- $101 \nmid n(n + 1)$. We prove the following claim.

  Claim 1. Suppose $101 \nmid n(n + 1)$, and let $j \in \{1, 2, \ldots, 100\}$. Then, there exists an integer in the interval $I_1 = \left[\frac{nj}{101}, \frac{(n+1)j}{101}\right)$ if and only if there does not exist an integer in the interval $I_2 = \left[\frac{n(101-j)}{101}, \frac{(n+1)(101-j)}{101}\right)$.

  Proof. Rewrite the interval $I_2 = \left[n - \frac{nj}{101}, n + 1 - \frac{(n+1)j}{101}\right)$. Let $a, b \in \{0, 1, 2, \ldots, 100\}$ be the reductions of nj and $(n + 1)j$ modulo 101, respectively. We have $a \neq b$ since $0 < j < 101$, and both a and b are nonzero since $101 \nmid n(n + 1)$. By definition, $\frac{nj+101-a}{101}$ is the smallest integer greater than $\frac{nj}{101}$, and $n - \frac{nj-a}{101}$ is the smallest integer greater than $n - \frac{nj}{101}$.
  If $a > b$, then $\frac{nj+101-a}{101} = \frac{(n+1)j-b}{101}$ is the only integer in I_1, and there are no integers in I_2. Analogously, if $a < b$, then $n - \frac{(nj-a)}{101} = n - \frac{(n+1)j-b}{101}$ is the only integer in I_2, and there are no integers in I_1. The claim follows. □

  By the above claim, exactly one integer j from each pair $\{i, 101 - i\}$ for $i \in \{1, 2, 3, \ldots, 50\}$ has the property property that some k satisfies $\left\lfloor \frac{101k}{j} \right\rfloor = n$. Thus, $f(n) = 50$.

10. Let

$$S = \sum_{k=0}^{2026} k\binom{2k}{k}2^k.$$

Compute the remainder when S is divided by 2027. (Note that 2027 is prime.)

Proposed by: Jacopo Rizzo, Pitchayut Saengrungkongka

Answer: $\boxed{289}$

Solution: We show, more generally, that for any prime $p > 2$,

$$\sum_{k=0}^{p-1} k\binom{2k}{k}2^k \equiv 4 \cdot (-7)^{\frac{p-3}{2}} \bmod p.$$

This implies with $p = 2027$,

$$S \equiv 4 \cdot (-7)^{\frac{2027-3}{2}} \equiv \boxed{289} \bmod 2027.$$

It suffices to only consider the sum from $k = 0$ to $\frac{p-1}{2}$ as all larger terms have $\binom{2k}{k}$ divisible by p.

Claim 1. Over $\mathbb{F}_p[x]$,

$$(1-4x)^{\frac{p-1}{2}} = \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k} x^k.$$

(Here, we take $\mathbb{F}_p[x]$ as the set of polynomials with coefficients taken modulo p.)

Proof. All polynomial equalities below are taken over $\mathbb{F}_p[x]$. First, by the Binomial Theorem,

$$(1-4x)^{\frac{p-1}{2}} = \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}(-4x)^k.$$

Each $\binom{\frac{p-1}{2}}{k}$ can then be reduced mod p as follows:

$$\binom{\frac{p-1}{2}}{k} = \frac{(\frac{p-1}{2}) \cdot \ldots \cdot (\frac{p-1}{2} - k + 1)}{k!} \equiv \frac{(-\frac{1}{2}) \cdot \ldots \cdot (\frac{1-2k}{2})}{k!} \equiv \frac{(-1)^k(2k-1)!!}{2^k \cdot k!} \equiv \left(-\frac{1}{4}\right)^k \binom{2k}{k} \bmod p.$$

The claim immediately follows. □

Now, take the formal derivative of both sides in the above identity to obtain

$$\frac{p-1}{2} \cdot (-4) \cdot (1-4x)^{\frac{p-3}{2}} = \sum_{k=1}^{\frac{p-1}{2}} \binom{2k}{k} kx^{k-1}.$$

In particular, plugging $x = 2$, we get

$$S = 2 \cdot \left(\frac{p-1}{2}\right) \cdot (-4) \cdot (1 - 4 \cdot 2)^{\frac{p-3}{2}},$$

thus $S \equiv 4 \cdot (-7)^{\frac{p-3}{2}} \bmod p$.

Remark. One can avoid using formal derivatives by noting that

$$k\binom{2k}{k} \equiv k\binom{-\frac{1}{2}}{k} \equiv -\frac{1}{2} \cdot \binom{-\frac{3}{2}}{k-1} \bmod p$$

and proceeding with the polynomial $(1-4x)^{\frac{p-3}{2}}$.