HMMT February 2026

February 14, 2026

Guts Round

1. **[5]** Let H, M, and T be (not necessarily distinct) digits such that H is nonzero and

$$\underline{HMMT} = \underline{HTM} \times \underline{HT}.$$

Compute the only possible value of the four-digit positive integer $\underline{HMMT}$.

Proposed by: Jackson Dryg, Jason Mao

Answer: $\boxed{1000}$

Solution: Note that

$$(H+1) \cdot 1000 > \underline{HMMT} = \underline{HTM} \times \underline{HT} \geq (H \cdot 100) \times (H \cdot 10) = H^2 \cdot 1000.$$

Since $H^2 > 2H - 1 \geq H + 1$ for $H \geq 2$, we must have $H = 1$. Then, the equation becomes

$$1000 + 110M + T = (100 + 10T + M) \cdot (10 + T) = 1000 + 200T + 10T^2 + 10M + MT.$$

The above is equivalent to

$$(100 - T)M = T(10T + 199). \tag{1}$$

Clearly, if $M = 0$, then T must be zero, and if $T = 0$, then M must be zero. This leads to the solution $(H, M, T) = (1, 0, 0)$, when the equation reads $1000 = 100 \cdot 10$.

We will show that no other solution exists. For the sake if contradiction, assume that $M \neq 0$ and $T \neq 0$ leads to a valid solution. Then, both sides of (1) are nonzero. We set $d = \gcd(100 - T, 10T + 199)$. Since $10T + 199 \mid (100 - T)M$, we must have $(10T + 199)/d \mid M$ and $(199 + 10T)/d \leq M$. Therefore,

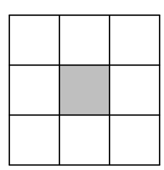
$$d \geq \frac{199 + 10T}{M} \geq \frac{199}{9} > 22.$$

On the other hand, we have that

$$d \mid 10 \cdot (100 - T) + (199 + 10T) = 1199.$$

Since $1199 = 11 \times 109$, the only possible choices for d are 109 and 1199. However, $d \mid 100 - T < 109$, which is a contradiction. Thus, the only solution is $(H, M, T) = (1, 0, 0)$, and so $\underline{HMMT} = \boxed{1000}$.

2. **[5]** Compute the number of ways to fill each of the outer 8 cells of a 3×3 grid with exactly one of the numbers 1, 2, and 3 such that the top row, bottom row, left column, and right column each contain no repeated numbers.



Proposed by: Sebastian Attlan

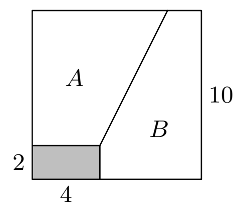
Answer: $\boxed{18}$

Solution: Notice that given distinct numbers in two corner cells on the same row or column, there is a unique choice for the number in the middle cell on the same row or column. Furthermore, every middle cell participates in a single row or column, so the only requirement for a number in a middle cell is to be different from the two corner cells on its row or column. Thus, it is sufficient to compute the number of ways to fill each of the 4 corner cells with exactly one of the numbers 1, 2, and 3 such that no two corner cells on the same row or column have the same number.

Let a, b, c, and d denote the numbers in the upper-left, upper-right, lower-left, and lower-right corner cells, respectively. If $a = c = x$, then there are three possible choices for x, two possible choices for $b \neq x$, and two possible choices for $d \neq x$. If $a \neq c$, then there are three are three possible choices for a, two possible choices for c, and $b, d \notin \{a, c\}$ are uniquely determined. Therefore, the total number of ways to fill the corner cells without violating the problem condition is

$$3 \cdot 2 \cdot 2 + 3 \cdot 2 = \boxed{18}.$$

3. **[5]** A rectangle with length 4 and height 2 is placed in the bottom left corner of a square with side length 10. Regions A and B are separated by a line segment drawn from the rectangle's upper right corner to some point on the boundary of the square. Given that regions A and B have the same perimeter, compute the positive difference between the areas of regions A and B.



Proposed by: Sebastian Attlan

Answer: $\boxed{4}$

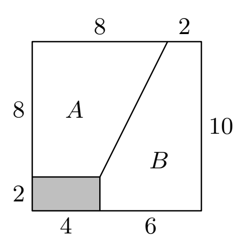
Solution: Let the line segment splitting A and B have length ℓ, and let it split the perimeter of the square that is not shared by the rectangle into lengths x and y, which are on the borders of A and B, respectively. Then $x + y = 8 + 10 + 10 + 6 = 34$.

The perimeter of A is $4 + x + \ell$, while the perimeter of B is $2 + y + \ell$. Then

$$4 + x + \ell = 2 + y + \ell,$$

so $y = x + 2$. Combining this with the condition $x + y = 34$ gives us $(x, y) = (16, 18)$.

Knowing this, the length of segments in the figure can be calculated as below:



Hence, the area of A (a trapezoid) is

$$\frac{4+8}{2} \cdot 8 = 48$$

while the area of B (a rectangle missing a triangle in its top left corner) is

$$6 \cdot 10 - \frac{4 \cdot 8}{2} = 44,$$

so their difference is $48 - 44 = \boxed{4}$.

4. **[5]** Over all nonnegative integers a, b, c, and d such that

$$ab + cd = 31 \quad \text{and} \quad ac + bd = 29,$$

compute the minimum possible value of $a + b + c + d$.

Proposed by: Derek Liu

Answer: $\boxed{17}$

Solution: Let $x = a + d$ and $y = b + c$. Adding the two given equations gives us $xy = 60$, and subtracting them gives us $(a - d)(b - c) = 2$. The latter tells us one of $a - d$ and $b - c$ is ± 1 and the other is ± 2; in particular, they have opposite parity.

Since x and $a - d$ differ by $2d$, they have the same parity. Likewise, y and $b - c$ have the same parity. Thus, x and y have opposite parity. Since $xy = 60$, the smallest possible sum of $a + b + c + d = x + y$ is $5 + 12 = \boxed{17}$.

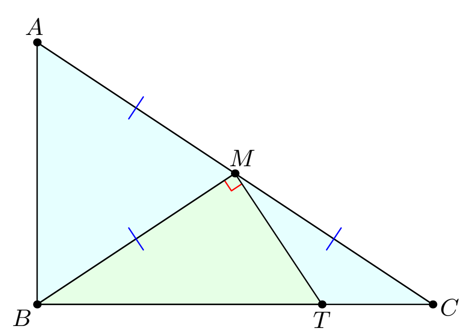
To show this is achievable, we note that $a + d = 5$, $a - d = 1$, $b + c = 12$, and $b - c = 2$ should give us a solution, and indeed $(a, b, c, d) = (3, 7, 5, 2)$ works.

5. **[6]** Let ABC be a right triangle with $\angle ABC = 90°$ and $AB < BC$. Let M be the midpoint of $\overline{AC}$. Let T be the unique point lying on the segment $\overline{BC}$ such that $\angle BMT = 90°$. Given that $AB = 5$ and $MT = 3$, compute CT.

Proposed by: Jason Mao

Answer: $\boxed{\frac{7}{\sqrt{11}} = \frac{7\sqrt{11}}{11}}$

Solution:



Since M is the midpoint of AC, $AM = MB = MC$. In particular, triangle MBC is isosceles, so $\angle MBT = \angle ACB$. We also have $\angle TMB = \angle ABC = 90°$, so $\triangle ABC \sim \triangle TMB$.

Let $AM = MB = MC = 5x$. The similarity yields

$$\frac{3}{5} = \frac{TB}{10x} \implies TB = 6x.$$

Pythagorean's Theorem on $\triangle MBT$ gives

$$(6x)^2 = (5x)^2 + 3^2 \implies x = \frac{3}{\sqrt{11}}.$$

Note that $BC = MB \cdot \frac{AB}{TM}$, so

$$CT = BC - TB = 5x \cdot \frac{5}{3} - 6x = \frac{7}{3}x = \boxed{\frac{7}{\sqrt{11}}}.$$

6. **[6]** Compute the largest positive integer n such that $n!$ divides

$$\binom{256}{128}^1 \cdot \binom{128}{64}^2 \cdot \binom{64}{32}^4 \cdot \binom{32}{16}^8 \cdot \binom{16}{8}^{16} \cdot \binom{8}{4}^{32} \cdot \binom{4}{2}^{64} \cdot \binom{2}{1}^{128}.$$

Proposed by: Andrew Brahms, Carlos Rodriguez

Answer: $\boxed{256}$

Solution: We can expand out the product into the following:

$$\frac{256!}{128!^2} \cdot \frac{128!^2}{64!^4} \cdot \frac{64!^4}{32!^8} \cdot \frac{32!^8}{16!^{16}} \cdot \frac{16!^{16}}{8!^{32}} \cdot \frac{8!^{32}}{4!^{64}} \cdot \frac{4!^{64}}{2!^{128}} \cdot \frac{2!^{128}}{1!^{256}}$$

After canceling, we see that the expression is equal to $256!$. Thus, the answer is $\boxed{256}$.

7. **[6]** A tromino is any connected figure constructed by joining 3 unit squares edge-to-edge. Compute the number of ways to tile a 2×6 rectangular grid with 4 nonoverlapping trominoes.

(Two tilings that differ by a rotation or reflection are considered distinct.)

Proposed by: Sebastian Attlan

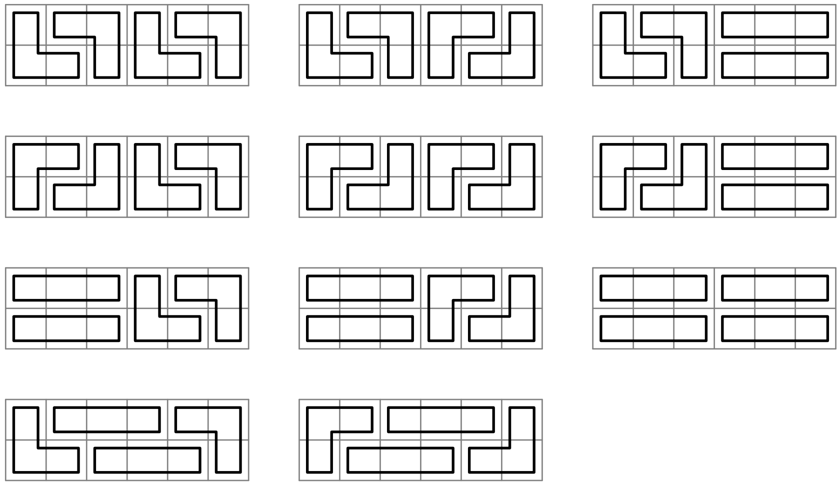
Answer: $\boxed{11}$

Solution: All trominos are either three squares joined in a line or joined in an L shape. Consider the four edges in the middle with no endpoints on the boundary of the grid. We can now perform casework on the intersections of trominos with these edges.

- If none of these edges are crossed by a tromino, then the trominos must form a chain tracing the boundary of the grid. There are 3 such tilings depending on where the top-leftmost tromino starts and ends.
- Suppose that at least one of these four middle edges are crossed by a tromino. (Assume that the long edges of the grid are horizontal.) If the edge in the 2nd or 5th column is crossed, then it leaves an isolated region of area 1 or 2, which can't be covered by a tromino. If the edge in the 3rd or 4th column is crossed, then we must be able to split the tiling into tilings of two 2×3 subgrids, each of which is covered by a chain tracing the boundary of the subgrid. There are 3 possibilities for each, giving us $3 \cdot 3 - 1 = 8$ possible tilings since one of those tilings overlaps with the first case above.

Then we have $3 + 8 = \boxed{11}$ tilings in total.

Here is the diagram that lists all 11 tilings.



8. **[6]** Let a_1, a_2, $\ldots$ be a sequence of positive integers such that $a_1 = 2$ and for all $n \geq 2$, it holds that a_n is the sum of a_{n-1} and the largest prime divisor of a_{n-1}. Compute the smallest integer greater than 2026 that appears in this sequence.

   Proposed by: Matthew Qian

   Answer: $\boxed{2068}$

   Solution: The first few terms of the sequence are

$$2, 4, 6, 9, 12, 15, 20, 25, 30, 35, 42, 49, \ldots$$

Notice that $2 \cdot 3 = 6$, $3 \cdot 5 = 15$, and $5 \cdot 7 = 35$ all appear in the sequence. This suggests the following more general fact.

Claim 1. Let p_1, p_2, $\ldots$ be the list of primes in increasing order. Then, for all $i \geq 1$, the value $p_i p_{i+1}$ appears in the sequence a_1, a_2, $\ldots$.

Proof. We proceed by induction on i. For the base case $i = 1$, we know $p_1 p_2 = 6$ appears in the sequence, as desired.

Now assume $a_k = p_i p_{i+1}$ for some k. The largest prime factor of a_k is p_{i+1}, so we have $a_{k+1} = (p_i + 1)p_{i+1}$. Continuing this reasoning, we get that

$$a_{k+1} = (p_i + 1)p_{i+1}, a_{k+2} = (p_i + 2)p_{i+1}, \ldots,$$

and this will continue until a term has a prime factor greater than p_{i+1}. But this occurs precisely when the term becomes $p_{i+2}p_{i+1}$, so the induction is complete. $\square$

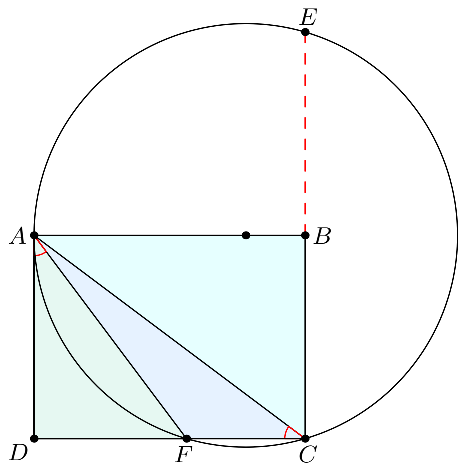
Using our claim, we know that the term $43 \cdot 47 = 2021$ appears in the sequence. Therefore, the smallest term greater than 2026 is the following term, which is $2021 + 47 = \boxed{2068}$.

9. **[7]** Let $ABCD$ be a rectangle. Let E be the reflection of C over B. The circumcircle of triangle ACE intersects line CD at a point $F \neq C$. Given that $AC = 8$ and $AF = 6$, compute the area of rectangle $ABCD$.

   Proposed by: Rishabh Das

   Answer: $\boxed{\frac{768}{25} = 30.72}$

   Solution:



Since B is the midpoint of $\overline{CE}$, and $AB \perp CE$, the circumcenter of triangle ACE lies on line AB. As $AB \perp AD$, line AD is tangent to the circumcircle of triangle ACE. Thus, $\angle FAD = \angle ACD$, and so $\triangle FAD \sim \triangle ACD$. Therefore,

$$\frac{3}{4} = \frac{AF}{CA} = \frac{AD}{CD}.$$

The above implies that $AD = 3x$ and $CD = 4x$ for some $x > 0$. Now, applying the Pythagorean theorem on triangle ACD yields

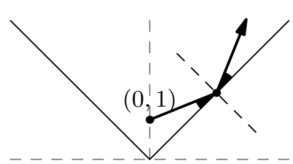
$$64 = AC^2 = AD^2 + CD^2 = (3x)^2 + (4x)^2 = 25x^2.$$

Finally, we compute that the area of $ABCD$ is

$$AD \cdot CD = (3x) \cdot (4x) = 12 \cdot \frac{64}{25} = \boxed{\frac{768}{25}}.$$

10. **[7]** Srinivas picks a uniformly random direction and shoots a laser starting at point $(0, 1)$ at his chosen direction. The laser bounces off the graph of $y = |x|$ whenever it makes contact. Compute the expected number of times the laser contacts the graph of $y = |x|$.

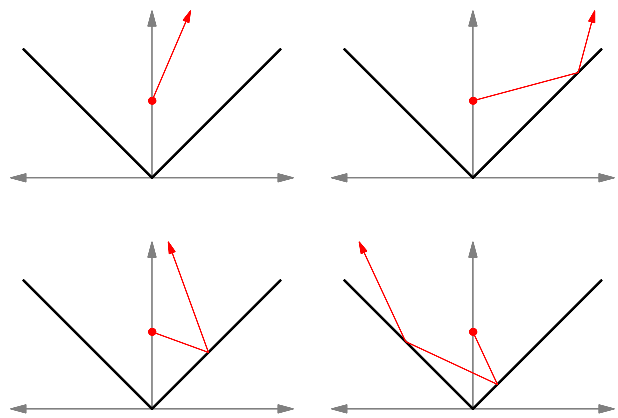
(When the laser bounces, the angle at which it arrives mirrors the angle at which it departs. See the diagram below.)



Proposed by: Sam EnMin Huang

Answer: $\boxed{1}$

Solution: Let θ be the angle of the laser relative to the x axis. Note that we may assume $-90^\circ \leq \theta \leq 90^\circ$, since by symmetry, the expected value of the number of bounces given that θ is on the right side of the y axis is the same as the expected value given that θ is on the left side.



- **If $45° \leq \theta \leq 90°$**, then the laser never hits the graph of $|x|$, because the angle of $|x|$ for positive x is $45° \leq \theta$. Thus, there are 0 bounces in this case.
- **If $0° \leq \theta < 45°$**, the laser bounces once, then stays on the same side of the y-axis but doesn't hit anything forever: after bouncing once, the new angle of the laser is

  $$45° < 90° - \theta \leq 90°.$$

  Thus, there is 1 bounce in this case.
- **If $-45° \leq \theta < 0°$**, the laser bounces once and eventually gets to the other side of the y axis but never hits anything: after bouncing once, the new angle of the laser is:

  $$90° < 90° - \theta \leq 135°.$$

  Thus, there is 1 bounce in this case.
- **If $-90° \leq \theta < -45°$**, the laser first bounces off the graph of $|x|$ where x is positive. The new angle is:

  $$135° < 90° - \theta \leq 180°.$$

  Thus, there are 2 bounces in this case.

Since all cases are equally likely, the expected value is

$$\frac{1}{4} \cdot (0 + 1 + 1 + 2) = \boxed{1}.$$

11. **[7]** Compute the number of ordered pairs (a, b) of positive integers such that $\operatorname{lcm}(a, b) + \gcd(a, b) = 2026$.

Proposed by: Marin Hristov Hristov

Answer: $\boxed{13}$

Solution: First, recall the identity $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$. By factoring out $\gcd(a, b)$, we can write $a = ka'$ and $b = kb'$, with $\gcd(a, b) = k$ and $\gcd(a', b') = 1$. In particular $\operatorname{lcm}(a, b) = ka'b'$.

The equation now rewrites to finding triples of positive integers (k, a', b') with $\gcd(a', b') = 1$, such that

$$\begin{aligned} ka'b' + k &= 2026 \\ \implies k(a'b' + 1) &= 2026. \end{aligned}$$

In particular, k is a divisor of 2026, so we do casework on k.

- If $k=1$, then $a'b'=2025$. Since a' and b' are coprime, every prime factor p dividing 2025 must go to exactly one of a' or b'. This gives 2 choices for every prime. For $2025 = 3^4 \cdot 5^2$, this gives $2^2 = 4$ options: $(1, 3^4 \cdot 5^2), (3^4, 5^2), (5^2, 3^4), (3^4 \cdot 5^2, 1)$.
- If $k=2$, then $a'b' = 1012 = 2^2 \cdot 11 \cdot 23$. By the same argument presented above, we have a total of $2^3 = 8$ options.
- If $k = 1013$, then $a'b' = 1$, and there is only one option.
- If $k = 2026$, then $a'b' = 0$, which is impossible.

In total, there are $4+8+1+0 = \boxed{13}$ pairs (a,b).

12. **[7]** Let a, b, c, and d be positive real numbers such that $ac = 100$ and $bd = 101$. Compute the largest possible value of

$$a^{\log_{10} b} \cdot b^{\log_{10} c} \cdot c^{\log_{10} d} \cdot d^{\log_{10} a}.$$

Proposed by: Carlos Rodriguez

Answer: $\boxed{101^2 = 10201}$

Solution 1: Let $f(a,b,c,d) = a^{\log_{10} b} b^{\log_{10} c} c^{\log_{10} d} d^{\log_{10} a}$. Taking log base 10 of $f(a,b,c,d)$ yields:

$$\begin{aligned}
\log_{10}(f(a,b,c,d)) &= \log_{10}(a^{\log_{10} b} b^{\log_{10} c} c^{\log_{10} d} d^{\log_{10} a}) \\
&= \log_{10} a \log_{10} b + \log_{10} b \log_{10} c + \log_{10} c \log_{10} d + \log_{10} d \log_{10} a \\
&= (\log_{10} a + \log_{10} c)(\log_{10} b + \log_{10} d) \\
&= \log_{10}(ac) \log_{10}(bd) \\
&= \log_{10}(100) \log_{10}(101) \\
&= 2 \log_{10}(101).
\end{aligned}$$

Therefore, $f(a,b,c,d) = 10^{2\log_{10}(101)} = 101^2 = \boxed{10201}$ for all positive real numbers a, b, c, and d with $ac = 100$ and $bd = 101$.

Solution 2: Define f as before. Substituting $d = 101/b$ and $c = 100/a$ directly into the expression $f(a,b,c,d) = a^{\log_{10} b} b^{\log_{10} c} c^{\log_{10} d} d^{\log_{10} a}$ yields:

$$\begin{aligned}
f(a,b,c,d) &= a^{\log_{10} b} b^{\log_{10}(100/a)} (100/a)^{\log_{10}(101/b)} (101/b)^{\log_{10} a} \\
&= a^{\log_{10} b - \log_{10}(101/b)} b^{\log_{10}(100/a) - \log_{10} a} 100^{\log_{10}(101/b)} 101^{\log_{10} a} \\
&= \left(a^{2\log_{10} b} b^{2\log_{10}(1/a)}\right) \cdot \left(a^{\log_{10}(1/101)} 101^{\log_{10} a}\right) \cdot \left(b^{\log_{10}(100)} 100^{\log_{10}(1/b)}\right) \cdot 100^{\log_{10}(101)} \\
&= 1 \cdot 1 \cdot 1 \cdot 100^{\log_{10}(101)} \\
&= 101^2 = \boxed{10201}.
\end{aligned}$$

13. **[9]** The *concatenation* of two base-10 numbers (possibly with leading 0s) is defined as the base-10 number formed by joining them together. For example, the concatenation of 1402 and 00213 is 140200213. Compute the number of 2026-digit multiples of 3 which **cannot** be expressed as the concatenation of two smaller multiples of 3 (possibly with leading 0s).

Proposed by: Justin Zhang, Sidarth Erat

Answer: $\boxed{18 \cdot 7^{2024}}$

Solution: Let n be a 2026-digit multiple of 3. Then n satisfies the desired condition if and only if for all m from 1 to 2025, the first m digits of n have sum not divisible by 3. To compute the number of such n, we will construct n one digit at a time.

- There are 6 ways to pick the first digit, as it can be any non-multiple of 3. (any digits except $0, 3, 6, 9$)
- For each $m = 2, 3, \ldots, 2025$, given that the sum of the first $m-1$ digits of n, called d, is not divisible by 3.
  - If $d \equiv 1 \pmod 3$. Then, the m^{th} digit of n can neither be 2, 5, nor 8, so there are only 7 remaining choices.
  - If $d \equiv 2 \pmod 3$. Then, the m^{th} digit of n can neither be 1, 4, nor 7, so there are only 7 remaining choices.

  In either case, there are only 7 ways to pick the m^{th} digit.
- To pick the last digit, not that the sum of all digits must be divisible by 3. If the sum of the first 2025 digits has remainder 1 when divided by 3, the last digit can only be $2, 5, 8$. Similarly, if the remainder is 2 instead, the last digit can only be $1, 4, 7$. In either case, there are 3 ways to choose the last digit.

Hence, the answer is

$$6 \cdot 7^{2024} \cdot 3 = \boxed{18 \cdot 7^{2024}}.$$

14. **[9]** There exists exactly one ordered pair of positive integers (m, n), both greater than 1, with the property that, when written out in base 10, $m \cdot n = \underline{ABCD}$ and $\binom{m}{n} = \underline{CDAB}$ for distinct nonzero digits A, B, C, and D. Compute $m + n$.

Proposed by: Andrew Brahms

Answer: $\boxed{136}$

Solution: We begin with the following crucial claim.

Claim 1. We have $n = m - 2$.

Proof. Note that $m > n$ since $\binom{m}{n} = \underline{CDAB} \geq 1111$. Furthermore, $mn = \underline{ABCD} \geq 1111$, so $m > \sqrt{1111} > 33$. We continue by showing that any pair (m, n) not satisfying $n = m - 2$ can't lead to a solution.

- We are given that $n \neq 1$.
- If $n = m - 1$, then $\binom{m}{n} = m$. Hence, $m = \underline{CDAB} \geq 1111$, and therefore

  $$10000 \geq \underline{ABCD} = mn = m(m-1) \geq 1111 \cdot 1110,$$

  which is clearly a contradiction.
- If $n \in \{2, 3\}$, then $m \geq \underline{ABCD}/3 \geq 1111/3 > 370$. However, $\underline{CDAB} = \binom{m}{n} \geq \binom{370}{2} > 10000$ clearly leads to a contradiction.
- If $4 \leq n \leq m - 4$, then $\underline{CDAB} = \binom{m}{n} \geq \binom{m}{4} \geq \binom{33}{4} > 10000$.
- Finally, we consider the case $n = m - 3$. We must have that

  $$\frac{m(m-4)(m-5)}{6} = \binom{m}{m-3} - m(m-3) = \underline{CDAB} - \underline{ABCD} = 99 \cdot (\underline{CD} - \underline{AB}).$$

  Therefore, $27 \mid m(m-4)(m-5)$. Since $m > 33$, and $m \leq 40$ (as $\binom{41}{3} = 10660 > \underline{CDAB}$), the numbers m, $m-4$, and $m-5$ lie in the interval $[29, 40]$. However, 3 can divide at most one of them, which leads to a contradiction as no integer in the interval $[29, 40]$ is divisible by 27. $\square$

Following Claim 1, we must have that $n = m - 2$. Let $\underline{AB} = k$, and $\underline{CD} = \ell$. Thus, $\binom{m}{n} = 100\ell + k$, and $mn = 100k + \ell$. Hence, we have that $\frac{m(m-1)}{2} = 100l + k$ and $m(m-2) = 100k + l$. Adding these equations yields

$$\frac{m(3m-5)}{2} = 101(k+l).$$

Therefore, $101 \mid m(3m-5)$. If $101 \mid m$, then either $m = 101$ which doesn't lead to a solution, or $m \geq 202$, in which case $m(m-2) \geq 202 \cdot 200 > 10000 > \underline{ABCD}$. Hence, $101 \mid 3m - 5$. If $m > 101$, then $m(m-2) \geq 102 \cdot 100 > 10000$, which leads to a contradiction. If $m \leq 101$, then

$$m \equiv 5/3 \equiv 69 \pmod{101} \implies m = 69.$$

Therefore, $m = 69$ and $n = 67$, which leads to a solution with $A = 4$, $B = 6$, $C = 2$, and $D = 3$ as $mn = 4623$ and $\binom{m}{n} = 2346$. Finally, we compute $m + n = 69 + 67 = \boxed{136}$.

Remark. Note that the above solution never uses the fact that the digits A, B, C, and D are distinct.

15. **[9]** Compute the number of ways to partition 2026 into the unordered sum of distinct positive integers, each of which is a power of 2 or a power of 6.

    Proposed by: Sebastian Attlan

    Answer: $\boxed{16}$

    Solution: The maximum possible power of 2 we can use is 2^{10}, while the maximum possible power of 6 we can use is 6^4. Therefore, we need to compute the number of subsets S of

    $$\{1, 2^1, 2^2, \ldots, 2^{10}, 6^1, 6^2, 6^3, 6^4\}$$

    whose sum is 2026.

    Given any subset B of $\{6^1, 6^2, 6^3, 6^4\}$ with sum $k \leq 2026$, there is a unique subset A of $\{1, 2^1, 2^2, \ldots, 2^{10}\}$ with sum $2026 - k$ corresponding to the binary representation of $2026 - k$. Hence, each possible B induces a unique possibility for S.

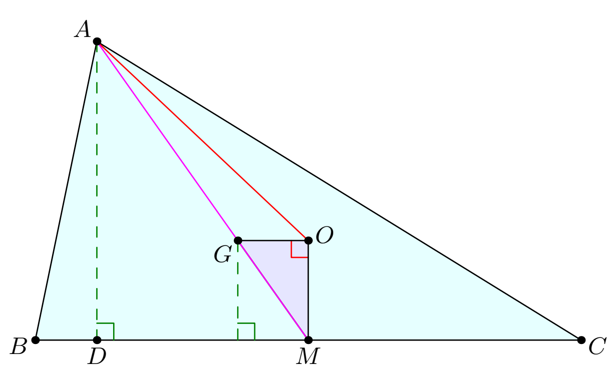
    Therefore, it suffices to compute the number of subsets B of $\{6^1, 6^2, 6^3, 6^4\}$ with sum at most 2026. But it is easy to see that all subsets satisfy this property, so the answer is simply $2^4 = \boxed{16}$.

16. **[9]** Let O and G be the circumcenter and centroid of triangle ABC, respectively, and let M be the midpoint of side $\overline{BC}$. Given that $OG = 1$, $OM = \sqrt{2}$, and $GM = \sqrt{3}$, compute the area of triangle ABC.

    Proposed by: Jason Mao

    Answer: $\boxed{3\sqrt{30} = \sqrt{270}}$

    Solution:



Observe that $\angle MOG$ is right, so lines OG and BC are parallel. Let D be the foot of the altitude from A to BC so since $AM = 3 \cdot GM$, we have $AD = 3 \cdot OM = 3\sqrt{2}$. We also note $MD = 3 \cdot OG = 3$.

This means letting r be the radius of the circumcircle of $\triangle ABC$, we see

$$OM^2 + BM^2 = r^2 = (AD - OM)^2 + (MD)^2 = 17.$$

Solving for BM gets $BM = \sqrt{15}$ so $BC = 2\sqrt{15}$. This means that our area is

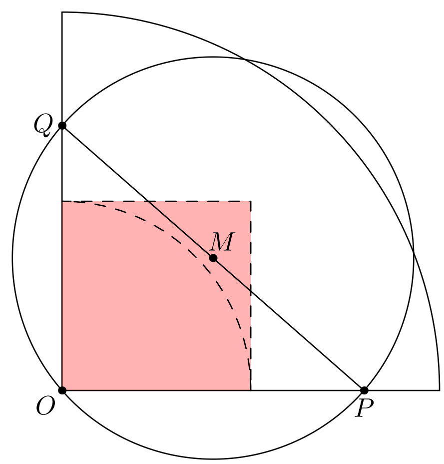
$$\frac{1}{2} \cdot \left(3\sqrt{2}\right) \cdot \left(2\sqrt{15}\right) = \boxed{3\sqrt{15}}.$$

17. **[11]** A point P is selected uniformly at random on one of the straight edges of a quarter circle, and another point Q is chosen independently and uniformly at random on the other straight edge. Compute the probability there exists a point A on the arc of the quarter circle such that $\angle PAQ$ is obtuse.

Proposed by: Rohan Bodke

Answer: $\boxed{1 - \frac{\pi}{4}}$

Solution:



Let M be the midpoint of PQ and O be the center of the quarter circle. Denote ω by the circle with diameter PQ. The region of points X for which $\angle PXQ$ is obtuse is the interior of ω. Thus, the point A exists if and only if ω intersects the arc of the quarter circle at two points, which only happens when OM plus the radius of ω is greater than the radius of the quarter circle. Because ω passes through M, the radius of ω is equal to OM. Thus, we need to find the probability that $2 \cdot OM$ is greater than the radius of the quarter circle.

To compute this probability, set the coordinate O to $(0, 0)$, ray OP to x-axis and OQ to y-axis. WLOG the radius of the quarter circle is 2. Then, $P = (x, 0)$ and $Q = (0, y)$ where x and y are random variable independently sampled uniformly from $[0, 2]$. Hence, the distribution of $M = (x/2, y/2)$ is uniform on the red unit square as shown in the image. We need to find the probability that $2OM > 2$, which happens when M is outside the circle of radius 1 centered at O. This is exactly the portion of the red unit square that does not lie on the unit circle centered at O, which is

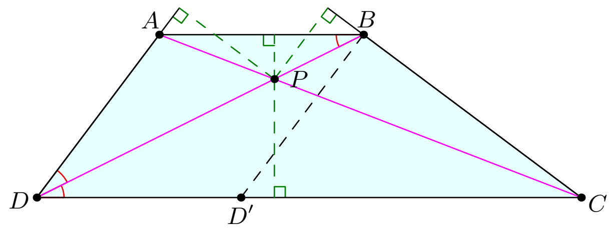
$$\frac{1 - \pi/4}{1} = \boxed{1 - \frac{\pi}{4}}.$$

18. **[11]** Let $ABCD$ be a trapezoid with side $\overline{AB}$ parallel to side $\overline{CD}$. Let P be the intersection of diagonals $\overline{AC}$ and $\overline{BD}$. Given that the distances from P to sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ are 3, 6, 8, and 8, respectively, compute the perimeter of $ABCD$.

Proposed by: Jason Mao

Answer: $\boxed{\frac{165}{2} = 82.5}$

Solution:



Let D' be the point such that $ABD'D$ is a parallelogram, and let $AB = 3x$. Then:

- Since $ABCD$ is a trapezoid, $\triangle PAB \sim \triangle PCD$, so similarity gives $CD = \frac{8}{3}AB = 8x$. Thus,
$$CD' = CD - DD' = 8x - 3x = 5x.$$
- Since P is equidistant from CD and DA, we have $\angle PDA = \angle PDC = \angle PBA$, so
$$BD' = AD = AB = 3x.$$
- Since $\triangle PAB \sim \triangle PCD$, we have $PA \cdot PD = PB \cdot PC$. Multiplying both sides by $\frac{1}{2}\sin\angle APD = \frac{1}{2}\sin\angle BPC$, we derive $[PAD] = [PBC]$. Using the distances from P to AD and BC,
$$\frac{1}{2} \cdot 8 \cdot (3x) = \frac{1}{2} \cdot 6 \cdot BC \implies BC = 4x.$$

Thus, BCD' has side lengths $3x$, $4x$, and $5x$, making it a right triangle of area $\frac{(3x)(4x)}{2} = 6x^2$. The height of the trapezoid is then the distance from B to CD', which is $\frac{2 \cdot 6x^2}{5x} = \frac{12x}{5}$. Hence,

$$[ADC] = \frac{1}{2} \cdot \frac{12x}{5} \cdot CD = \frac{1}{2} \cdot \frac{12x}{5} \cdot 8x = \frac{48x^2}{5}.$$

Using the distances from P to AD and CD, we also know

$$[ADC] = [APD] + [DPC] = \frac{1}{2} \cdot 8 \cdot 3x + \frac{1}{2} \cdot 8 \cdot 8x = 44x.$$

Thus,

$$\frac{48x^2}{5} = 44x \implies x = \frac{44 \cdot 5}{48} = \frac{55}{12},$$

and the perimeter of $ABCD$ is

$$3x + 4x + 8x + 3x = 18x = \boxed{\frac{165}{2}}.$$

19. **[11]** Compute the smallest positive integer n for which n has exactly 10 positive integer divisors and $n+1$ has exactly 9 positive integer divisors.

Proposed by: Derek Liu

Answer: $\boxed{6723}$

Solution: Since $n+1$ has an odd number of divisors, it must be a perfect square. Let $k = \sqrt{n+1}$, so $n = (k-1)(k+1)$. For n to have 10 divisors, it must be of the form p^9 or p^4q for primes p and q.

First assume $k-1$ and $k+1$ share a prime factor. As they differ by 2, this prime must be 2, so $2^2 \mid n$. Thus $n = 2^4q$ for some prime q (or 2^9, but 2^9+1 is not a square). Exactly one of $k-1$ and $k+1$ is divisible by 4, so $k-1$ and $k+1$ must be 2^3 and $2q$ in some order (or 2 and 2^3q, which clearly differ by more than 2). Thus $k=3$ or $k=5$. Neither of these work, as $k^2 = n+1$ must have 9 divisors.

Thus, $k-1$ and $k+1$ are relatively prime, which implies their product cannot be p^9, so they are p^4 and q in some order, and $q = p^4 \pm 2$. The next-smallest possible p is $p = 3$, with $q = 79$ or 83, and thus $k = 80$ or 82. While 80^2 has more than 9 divisors, 82^2 has exactly 9, making our answer $82^2 - 1 = \boxed{6723}$.

20. **[11]** Derek is at the front of a line, with six clones named Derek #1, Derek #2, Derek #3, Derek #4, Derek #5, and Derek #6 standing behind him in a uniformly random order. For all positive integers k between 1 and 6, inclusive, on the k^{th} minute from now, Derek #k will swap positions with whoever is standing directly in front of him in the line. Compute the probability that after 6 minutes, Derek is still at the front of the line.

Proposed by: Srinivas Arun

Answer: $\boxed{\frac{53}{144}}$

Solution 1: Let a_n be the answer to the problem if there are n total people in the line. We wish to find a_7.

The key observation is that once Derek #1 swaps positions with the person in front of him, the actions of all people standing behind Derek #1 become irrelevant, as they cannot swap with anyone in front of Derek #1. Using this observation, we compute a_n recursively as follows:

- There is a $\frac{1}{n-1}$ chance that Derek #1 starts at the 2nd position. In this case, since Derek is not at the front after the first swap and never moves forward, the probability Derek remains at the front is 0.
- There is a $\frac{1}{n-1}$ chance that Derek #1 stars at the 3rd position. In this case, after Derek #1 swaps, we can ignore him and everyone behind him, which leaves one person, so the probability Derek remains at the front is a_1.
- There is a $\frac{1}{n-1}$ chance that Derek #1 starts at the 4th position. In this case, after Derek #1 swaps, we can ignore him and everyone behind him, which leaves two people, so the probability Derek remains at the front is a_2.
- ...
- There is a $\frac{1}{n-1}$ chance that Derek #1 starts at the last position. In this case, after Derek #1 swaps, we can ignore him and everyone behind him, which leaves $n-2$ people, so the probability Derek remains at the front is a_{n-2}.

Thus, we get $a_n = \frac{1}{n-1}(a_1 + \cdots + a_{n-2})$. Using $a_1 = 1$ and $a_2 = 0$, we can recursively compute

$$\begin{aligned}
a_3 &= \tfrac{1}{2}(1) &&= \tfrac{1}{2}, \\
a_4 &= \tfrac{1}{3}(1+0) &&= \tfrac{1}{3}, \\
a_5 &= \tfrac{1}{4}\left(1+0+\tfrac{1}{2}\right) &&= \tfrac{3}{8}, \\
a_6 &= \tfrac{1}{5}\left(1+0+\tfrac{1}{2}+\tfrac{1}{3}\right) &&= \tfrac{11}{30}, \\
a_7 &= \tfrac{1}{6}\left(1+0+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{3}{8}\right) &&= \boxed{\tfrac{53}{144}}.
\end{aligned}$$

(Computing a_6 can be skipped.)

Solution 2: We say Derek#i has *label* i, and let j be the largest positive integer for which the first j clones (not including Derek himself) have labels in descending order.

Claim 1. Derek stays at the front of the line if and only if j is even.

Proof. We proceed with induction on j.

Base case: $j = 1$. The clone right behind Derek will swap with Derek before anyone else can swap with that clone, ensuring Derek ends up not in the front.

Induction Step: Assume the claim holds for $j - 1$. Let D denote whichever clone is right behind Derek, and observe that there are $j - 1$ clones after D with labels in descending order who all swap before D, with the last of them swapping before anyone else can swap with him (since j is maximal). After those $j - 1$ clones have made their swaps, by the induction hypothesis,

- if j is even, $j - 1$ is odd, so D is no longer right behind Derek. Whoever is right behind Derek has already made their swap, so Derek will remain in the front of the line.
- if j is odd, $j - 1$ is even, so D is still right behind Derek and will swap with him, ensuring Derek ends up not in the front.

This completes the proof. $\square$

The probability that j is at least m is the probability that the first m people are in descending order, or $1/m!$. Thus, for $m < 6$, the probability that $j = m$ is $1/m! - 1/(m+1)!$. We conclude the answer is

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} = \boxed{\frac{53}{144}}.$$

Remark. Remark: A generalized version of this solution shows that the desired probability with n clones equals the probability that a uniformly random permutation on n elements is a derangement, i.e., no element ends up in its original spot.

21. **[12]** Compute the largest possible value of

$$\gcd\left(\binom{n}{3} - 1, \binom{n}{4} - 1\right)$$

as n ranges through all positive integers greater than 3.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{102}$

Solution: We first prove the following claim.

Claim 1. Let p be an integer-coefficient polynomial and d be a positive integer, and define the function $f(n) = p(n)/d$. Let n_1 and n_2 be integers, q be a prime, and d' be the largest power of q that divides d. if $n_1 \equiv n_2 \pmod{qd'}$, then $f(n_1) \equiv f(n_2) \pmod{q}$.

Proof. Since $qd' \mid n_2 - n_1$, we know $qd' \mid p(n_2) - p(n_1)$. Since d/d' is not divisible by q, we conclude

$$qd' \,\middle|\, \frac{d'}{d}(p(n_2) - p(n_1)) \implies q \,\middle|\, \frac{1}{d}(p(n_2) - p(n_1)) = f(n_2) - f(n_1). \qquad \square$$

In particular, both $f_3(n) = \binom{n}{3} - 1$ and $f_4(n) = \binom{n}{4} - 1$ can be written in this form $p(n)/d$ with $d = 6$ and $d = 24$, respectively. With that in mind, we note that a gcd of 102 is achievable by $n = 823$ (or more generally, $n \equiv 823 \pmod{2448}$ where $2448 = 2^4 \cdot 3^2 \cdot 17$). This works because

- the gcd is divisible by 2 since $n \equiv 7 \pmod{2 \cdot 2^3}$ and 2 divides both $\binom{7}{3} - 1 = 34$ and $\binom{7}{4} - 1 = 34$,
- the gcd is divisible by 3 since $n \equiv 4 \pmod{3 \cdot 3^1}$ and 3 divides both $\binom{4}{3} - 1 = 3$ and $\binom{4}{4} - 1 = 0$, and
- the gcd is divisible by 17 since $n \equiv 7 \pmod{17 \cdot 17^0}$ and 17 divides both $\binom{7}{3} - 1 = 34$ and $\binom{7}{4} - 1 = 34$.

(The exact value of 823 need not be calculated, as it exists by Chinese Remainder Theorem.)

We now prove that the gcd has to be at most 102. Let $d = \gcd\left(\binom{n}{3} - 1, \binom{n}{4} - 1\right)$. We first show that d divides 204. To do this, note that d divides

$$\begin{aligned}
&(n-3)\left(\binom{n}{3} - 1\right) - 4 \cdot \left(\binom{n}{4} - 1\right) \\
&= (n-3)\left(\frac{n(n-1)(n-2)}{6} - 1\right) - 4 \cdot \left(\frac{n(n-1)(n-2)(n-3)}{24} - 1\right) \\
&= (n-3) - 4 = n - 7.
\end{aligned}$$

Since d also divides

$$6\left(\binom{n}{3} - 1\right) = n(n-1)(n-2) - 6 = (n-7)(n^2 + 4n) + 204,$$

we get that $d \mid 204$.

It remains to show that d cannot be divisible by 4. Assume for sake of contradiction that 4 divides both $\binom{n}{3} - 1$ and $\binom{n}{4} - 1$. Since 4 divides $\binom{n}{3} - 1$, we have that

$$n(n-1)(n-2) \equiv 6 \pmod{16},$$

and since 4 divides $\binom{n}{4} - 1$, we have that

$$n(n-1)(n-2)(n-3) \equiv 24 \pmod{64}.$$

In particular, $\nu_2(n(n-1)(n-2)) = 1$ and $\nu_2(n(n-1)(n-2)(n-3)) = 3$, so $\nu_2(n-3) = 2$, which means $n \equiv 7 \pmod{8}$. However, this means

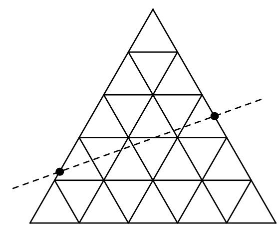
$$n(n-1)(n-2) \equiv 7 \cdot (7-1) \cdot (7-2) \equiv 2 \pmod{8},$$

contradicting $n(n-1)(n-2) \equiv 6 \pmod{16}$.

Hence, $4 \nmid d$, As $d \mid 204$, we conclude $d \leq \boxed{102}$.

22. **[12]** An equilateral triangle-shaped cake of side length 5 is cut into 25 unit equilateral triangle pieces. Jacob selects two distinct edges of the cake, then picks one point independently and uniformly at random on each of the two selected edges. He cuts along the line through these two points. Compute the expected number of pieces of cake after all cuts.

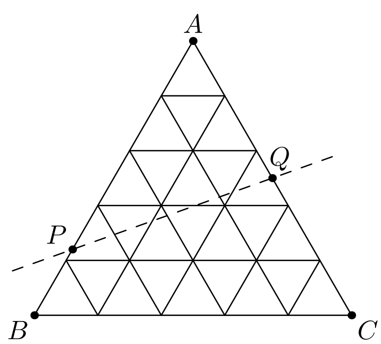
Below is an example of the process, with the dots being the selected points and the dashed line being Jacob's cut. This cut results in 32 pieces.



Proposed by: Sebastian Attlan

Answer: $\boxed{\frac{158}{5} = 31.6}$

Solution: It suffices to consider how many of the original 25 pieces Jacob's cut intersects, as the total number of pieces will be 25 plus this. Label the vertices of the cake ABC and the cut points P and Q such that P is on edge AB and Q is on edge AC. Let $AB = 5$.



With probability 1, the cut does not pass through any vertex of any original piece, so we assume this is the case.

Claim 1. The number of pieces Jacob's cut intersects is $2\max(\lceil AP \rceil, \lceil AQ \rceil) - 1$.

Proof. Let $p = \lceil AP \rceil$ and $q = \lceil AQ \rceil$. Jacob's cut passes through $|p-q|$ edges parallel to BC, $p-1$ edges parallel to CA, and $q-1$ edges parallel to AB. Because of our assumption, the number of pieces the cut passes through is

$$1 + |p-q| + (p-1) + (q-1) = (p+q) + |p-q| - 1 = 2\max(p,q) - 1. \qquad \square$$

As an example, in the diagram above, p and q are 4 and 3, and the cut indeed passes through $2 \cdot 4 - 1 = 7$ pieces.

Since AP and AQ are independent uniformly random real numbers from 0 to 5, we know $p = \lceil AP \rceil$ and $q = \lceil AQ \rceil$ are independently and uniformly selected from $\{1,2,3,4,5\}$. For positive integers $m \le 5$, the probability that p and q are both at most m is $\frac{m^2}{25}$, so the probability their maximum is m is $\frac{m^2}{25} - \frac{(m-1)^2}{25} = \frac{2m-1}{25}$. Thus, the answer is

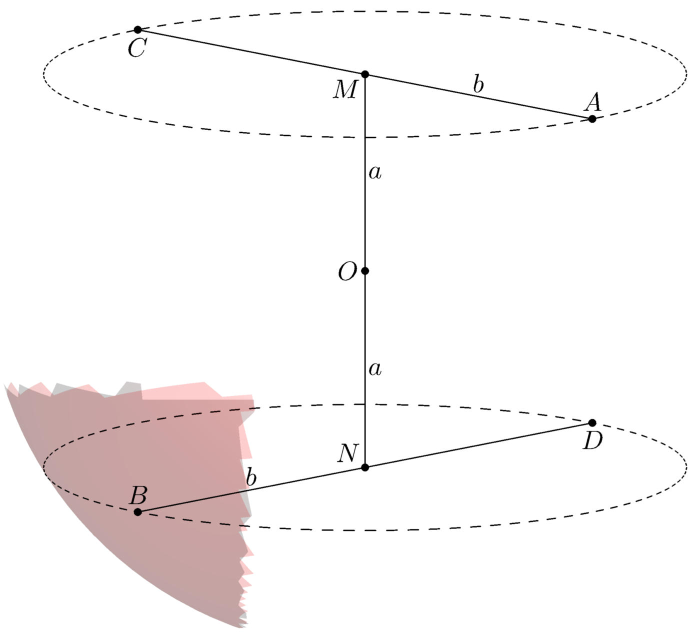
$$25 + \sum_{m=1}^{5} \frac{2m-1}{25}(2m-1) = 25 + \frac{1^2+3^2+5^2+7^2+9^2}{25} = 25 + \frac{165}{25} = \boxed{\frac{158}{5}}.$$

23. **[12]** Let Γ be a sphere of radius 5. Let A, B, C, and D be points on Γ such that $AB = BC = CD = DA = 8$ and $\angle ABC = \angle BCD = \angle CDA = \angle DAB$. Compute AC.

Proposed by: Jason Mao

Answer: $\boxed{6\sqrt{2} = \sqrt{72}}$

Solution:



Let O be the center of Γ, and let M and N be the midpoints of AC and BD, respectively. Then MN is perpendicular to both AC and BD, and O is the midpoint of M. Let $a = OM$ and $b = MA = NB$. By Pythagoras,

$$a^2 + b^2 = OM^2 + MA^2 = OA^2 = 25$$

and

$$b^2 + (2a)^2 + b^2 = AM^2 + MN^2 + NB^2 = 64$$

. Solving for a^2 and b^2, we get $b = \sqrt{18}$, so $AC = 2b = \boxed{6\sqrt{2}}$.

24. **[12]** Two mice and 100 pieces of cheese are uniformly and independently placed at random on the boundary of a circle. Each mouse walks to the piece of cheese closest to it, with ties broken independently at random. Compute the probability that the two mice walk to the same piece of cheese.

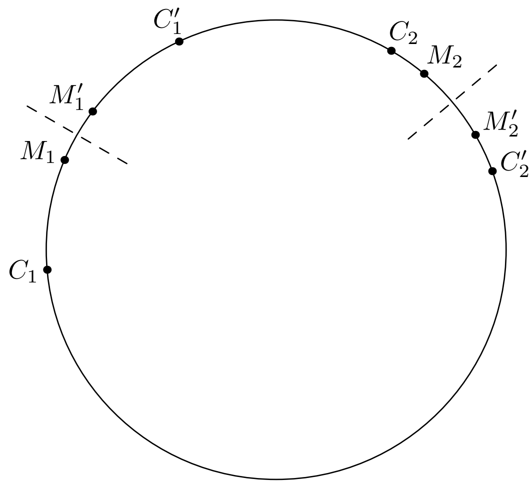
Proposed by: Sebastian Attlan

Answer: $\boxed{\frac{3}{202}}$

Solution 1: The key claim is the following.

Claim 1. We may assume without loss of generality that each mouse picks the nearest piece of cheese clockwise of it with probability $\frac{1}{2}$ and the nearest piece of cheese counterclockwise of it with probability $\frac{1}{2}$.

Proof. Label the mice M_1 and M_2. Let the nearest pieces of cheese clockwise and counterclockwise of M_1 be C_1 and C_1', and let the nearest pieces of cheese clockwise and counterclockwise of M_2 be C_2 and C_2'. Let M_1' lie on arc C_1C_1' such that arcs C_1M_1 and $M_1'C_1'$ have the same measure, and let M_2' lie on arc C_2C_2' such that arcs C_2M_2 and $M_2'C_2$ have the same measure.



Notice that the maps induced by sending M_1 to M_1' and/or sending M_2 to M_2' are bijections on the set of all possible placements of mice and cheese. Therefore, the mice positions (M_1, M_2), (M_1', M_2), (M_1, M_2'), and (M_1', M_2) are all equally likely. Since exactly one of M_1 and M_1' is closer to C_1 than C_1', and exactly one of M_2 and M_2' is closer to C_2 than C_2', the claim follows. $\square$

Notice that all orderings of the two mice and the 100 pieces of cheese along the circle are equally likely. Moreover, if there is more than one piece of cheese between the mice, then it is impossible for the mice to pick the same piece of cheese. Therefore, we have two cases.

- **Case 1: There are no pieces of cheese between the mice.**

  There are 102 possible orderings where the mice are consecutive, so the probability of this case is $\frac{102}{\binom{102}{2}} = \frac{2}{101}$. Now focus on the pieces of cheese surrounding the mice: $C_1M_1M_2C_2$. Using our key claim, there is a $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ probability that both mice pick C_1, and (similarly) a $\frac{1}{4}$ probability that both mice pick C_2. Thus, the overall contribution of this case is $\frac{2}{101} \cdot \frac{1}{2} = \frac{1}{101}$.

- **Case 2: There is exactly one piece of cheese between the mice.**

  There are 102 possible orderings where the mice are separated by one piece of cheese, so the probability of this case is $\frac{102}{\binom{102}{2}} = \frac{2}{101}$. Now focus on the pieces of cheese surrounding the mice: $C_1M_1C_2M_2C_3$. Using our key claim, there is a $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ probability that both mice pick C_2. Thus, the overall contribution of this case is $\frac{2}{101} \cdot \frac{1}{4} = \frac{1}{202}$.

Our answer is $\frac{1}{101} + \frac{1}{202} = \boxed{\frac{3}{202}}$.

Solution 2: Fix the position of the cheese that both mice come to, and then multiply by 100 later. We have two cases. Suppose that this cheese is distance x away from the first mouse and y away from the second mouse. Remembering to multiply by 2 later, we assume that the first mice is clockwise of the target cheese (as opposed to counterclockwise). Thus, we have to multiply by 200 later.

Case 1: The two mice are on opposite sides of the cheese

Then all other 99 cheese must be in a segment of length $(1-2x-2y)$. In particular, the probability that this happens (conditioned on fixed x and y) is $(1-2x-2y)^{99}$ if $x+y \le \frac{1}{2}$ and 0 otherwise. Therefore, by integrating across all x and y, the answer is

$$I = \int_{x=0}^{1/2} \int_{y=0}^{1/2-x} (1-2x-2y)^{99}\, dy\, dx.$$

Evaluating this integral, we get that

$$\begin{aligned}
I &= \int_{t=0}^{1/2} \int_{x=0}^{t} (1-2t)^{99}\, dx\, dt && \text{(substitute } t = x+y\text{)} \\
&= \int_{t=0}^{1/2} t(1-2t)^{99}\, dt \\
&= \frac{1}{2}\int_{u=0}^{1} u^{99} \cdot \frac{1-u}{2}\, du && \text{(substitute } u = 1-2t\text{)} \\
&= \frac{1}{4}\int_{0}^{1} u^{99} - u^{100}\, du \\
&= \frac{1}{4}\left(\frac{u^{100}}{100} - \frac{u^{101}}{101}\bigg|_{u=0}^{1}\right) \\
&= \frac{1}{4}\left(\frac{1}{100} - \frac{1}{101}\right) = \frac{1}{40400},
\end{aligned}$$

so the probbability in this case is $\frac{1}{40400}$.

Case 2: The two mice are on the same side of the cheese

Without loss of generality, let $x < y$ (and multiply by 2 later). Then the remaining 99 cheeses must be at least y away from the second mouse. Thus, all the cheeses must be in a segment of length $1-2y$. Therefore, the probability in this case is

$$\int_{y=0}^{1/2} \int_{x=0}^{y} (1-2y)^{99}\, dx\, dy.$$

Evaluating the inner layer, this is equal to

$$\int_{y=0}^{1/2} y(1-2y)^{99}\, dy,$$

which is an integral that also appears above, so it is $\frac{1}{40400}$. Remembering to multiply by 2, the answer to this case is $\frac{2}{40400}$.

Finally, adding two cases and remembering to multiply by $2 \cdot 100$, the final answer is

$$2 \cdot 100 \cdot \left(\frac{1}{40400} + \frac{2}{40400}\right) = \boxed{\frac{3}{202}}.$$

25. **[14]** Let $p(x)$ be the unique polynomial of degree at most 8 and with rational coefficients such that $p(\sqrt[3]{2}+\sqrt[3]{3}) = \sqrt[3]{6}$. Compute $p(1)$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{42}{25} = 1.68}$

Solution: Let $a = \sqrt[3]{2} + \sqrt[3]{3}$. We have

$$a^3 = 5 + 3\sqrt[3]{6}a \implies \sqrt[3]{6} = \frac{a^2}{3} - \frac{5}{3a}.$$

Now we just need to invert a. To do this, we find the minimal polynomial of a:

$$(a^3 - 5)^3 = 27 \cdot 6 \cdot a^3 \implies a^9 - 15a^6 - 87a^3 - 125 = 0,$$

which means that $\frac{1}{a} = \frac{a^8 - 15a^5 - 87a^2}{125}$, so

$$\sqrt[3]{6} = \frac{1}{3}\left(a^2 - \frac{a^8 - 15a^5 - 87a^2}{25}\right) = \frac{-a^8 + 15a^5 + 112a^2}{75}.$$

Thus, we conclude that

$$p(x) = \frac{-x^8 + 15x^5 + 112x^2}{75},$$

which implies that

$$p(1) = \frac{126}{75} = \boxed{\frac{42}{25}}.$$

26. **[14]** Marin is taking a random walk on a line. For each integer n ranging from 1 to 10, inclusive and in order, Marin takes a step of length F_n either left or right with equal probability, where F_n is the n^{th} Fibonacci number. Compute Marin's expected distance from his starting point.

(The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and the recurrence $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.)

Proposed by: Derek Liu

Answer: $\boxed{\frac{3741}{64}}$

Solution: Let d_n denote the expected distance after n steps. We will derive a recurrence for d_n.

Observe that after $n-1$ steps, if Bob is $x \leq F_n$ away from his starting position, his n^{th} step makes this distance either $F_n + x$ or $F_n - x$, with the average being F_n. If $x > F_n$, the two possible distances are $F_n + x$ and $x - F_n$, with average x instead. Thus, d_n is the expected value of $\max(F_n, x)$.

We claim that if $x > F_n$, then the $(n-2)^{\text{th}}$ and $(n-1)^{\text{th}}$ steps must be in the same direction. Indeed, otherwise the distance is upper-bounded by

$$F_{n-1} - F_{n-2} + F_{n-3} + \cdots + F_1 = F_{n-3} + (F_{n-1} - 1) < F_n.$$

The $(n-2)^{\text{th}}$ and $(n-1)^{\text{th}}$ steps are in the same direction with probability $1/2$. Given that this is the case, the expected value of $|x - F_n|$ is d_{n-3}, as the last two steps combined cover a distance of F_n. Note that $x - F_n$ is just as likely to be some value y as it is to be $-y$, so the expected value of $\max(x, F_n)$ in this case is

$$F_n + \max(x - F_n, 0) = F_n + \frac{1}{2}\mathbb{E}[|x - F_n|] = F_n + \frac{1}{2}d_{n-3}.$$

Recall that $x > F_n$ with probability $1/2$, so

$$d_n = \frac{1}{2}(F_n) + \frac{1}{2}\left(F_n + \frac{1}{2}d_{n-3}\right) = F_n + \frac{1}{4}d_{n-3}.$$

We conclude that

$$\begin{aligned}
d_{10} &= F_{10} + \frac{1}{4}d_7 \\
&= F_{10} + \frac{1}{4}F_7 + \frac{1}{4^2}d_4 \\
&= F_{10} + \frac{1}{4}F_7 + \frac{1}{4^2}F_4 + \frac{1}{4^3}d_1 \\
&= 55 + \frac{13}{4} + \frac{3}{16} + \frac{1}{64} \\
&= \boxed{\frac{3741}{64}}.
\end{aligned}$$

27. **[14]** Let a, b, and c be positive real numbers such that

$$\begin{aligned}
\sqrt{ab+1} + \sqrt{ca+1} &= 2a, \\
\sqrt{bc+1} + \sqrt{ab+1} &= 3b, \\
\sqrt{ca+1} + \sqrt{bc+1} &= 5c.
\end{aligned}$$

Compute a.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{14}{3\sqrt{11}} = \frac{14\sqrt{11}}{33}}$

Solution: Subtracting the second equation from the first equation gives

$$\sqrt{ca+1} - \sqrt{bc+1} = 2a - 3b. \tag{2}$$

Multiplying (2) by the third equation yields

$$ca - bc = (\sqrt{ca+1} + \sqrt{bc+1})(\sqrt{ca+1} - \sqrt{bc+1}) = 5c \cdot (2a - 3b).$$

Since $c > 0$, the above implies $a - b = 10a - 15b$, or, equivalently, $a/b = 14/9$. Similarly, we get $b/c = 9/5$. Therefore, we can write $a = 14t$, $b = 9t$, and $c = 5t$ for a positive real number t. Plugging $(a, b, c) = (14t, 9t, 5t)$ into the first equation gives

$$\sqrt{126t^2+1} + \sqrt{70t^2+1} = 28t. \tag{3}$$

Plugging $(a, b, c) = (14t, 9t, 5t)$ into (2) yields

$$\sqrt{126t^2+1} - \sqrt{70t^2+1} = 2t. \tag{4}$$

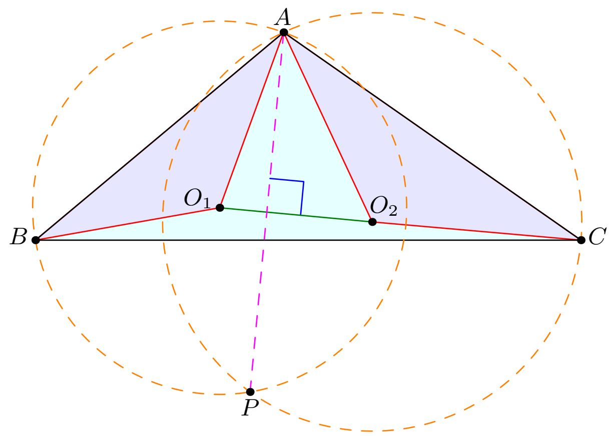
Adding (3) and (4) together gives $\sqrt{126t^2+1} = 15t$. Squaring both sides simplifies the equation to $99t^2 = 1$, whence $t = 1/\sqrt{99}$ as $t > 0$. Finally, $a = 14t = \boxed{\frac{14}{3\sqrt{11}}}$.

28. **[14]** Let ABC be a triangle such that $\angle BAC = 105^\circ$, $AB = 12\sqrt{2}$, and $AC = 17$. Let P be a point such that P and A lie on different sides of line BC, and $\angle APB = \angle APC = 60^\circ$. Compute AP.

Proposed by: Sarunyu Thongjarast

Answer: $\boxed{\frac{136\sqrt{3}}{13}}$

Solution:



Let O_1 and O_2 be the circumcenters of triangles ABP and ACP, respectively. Then AO_1B and AO_2C are isosceles triangles with $\angle AO_1B = \angle AO_2C = 2 \cdot 60^\circ = 120^\circ$. Thus,

$$AO_1 = \frac{1}{\sqrt{3}} \cdot AB = \frac{12\sqrt{2}}{\sqrt{3}}$$

and

$$AO_2 = \frac{1}{\sqrt{3}} \cdot AC = \frac{17}{\sqrt{3}}.$$

Furthermore,

$$\angle O_1AO_2 = \angle BAC - \angle BAO_1 - \angle O_2AC = 105^\circ - 30^\circ - 30^\circ = 45^\circ,$$

so

$$[AO_1O_2] = \frac{1}{2}(AO_1)(AO_2)\sin\angle O_1AO_2 = \frac{1}{2}\left(\frac{12\sqrt{2}}{\sqrt{3}}\right)\left(\frac{17}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{2}}\right) = 34.$$

By law of cosines on triangle AO_1O_2,

$$O_1O_2 = \sqrt{\left(\frac{12\sqrt{2}}{\sqrt{3}}\right)^2 + \left(\frac{17}{\sqrt{3}}\right)^2 - 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{12\sqrt{2}}{\sqrt{3}}\right)\left(\frac{17}{\sqrt{3}}\right)} = \frac{13}{\sqrt{3}}.$$

It follows that the distances from A to O_1O_2 is

$$2 \cdot \frac{[AO_1O_2]}{O_1O_2} = 2 \cdot \frac{34}{13/\sqrt{3}} = \frac{68\sqrt{3}}{13}.$$

Since $O_1A = O_1P$ and $O_2A = O_2P$, we know P is the reflection of A over line O_1O_2. Thus, AP is twice the distance from A to O_1O_2, or $\boxed{\frac{136\sqrt{3}}{13}}$.

29. **[16]** Compute

$$\sum_{k=1}^{4004} \gcd(k, 4004)\cos\left(\frac{\pi k}{2002}\right).$$

Proposed by: Jacopo Rizzo

Answer: $\boxed{1440}$

Solution 1: More generally, we claim that

$$\sum_{k=1}^{n} \gcd(k,n)\cos\left(\frac{2\pi k}{n}\right) = \varphi(n)$$

from which plugging in $n = 4004$ gives $\boxed{1440}$.

To prove the claim, we first note that the sum is the real part of

$$\sum_{k=0}^{n} \gcd(k,n)\zeta^k,$$

where $\zeta = e^{2\pi i/n}$ is the n-th root of unity. To evaluate this, we use $\sum_{d|n}\varphi(d) = n$ to get that

$$\begin{aligned}\sum_{k=1}^{n} \gcd(k,n)\zeta^k &= \sum_{k=1}^{n}\sum_{d|\gcd(k,n)} \varphi(d)\zeta^k \\ &= \sum_{k=1}^{n}\sum_{d|k \text{ and } d|n} \varphi(d)\zeta^k \\ &= \sum_{d|n}\varphi(d)\sum_{t=1}^{\frac{n}{d}}\zeta^{dt}.\end{aligned}$$

If $d = n$, then the inner sum is 1. Otherwise, by geometric series, the inner sum is

$$\frac{\zeta(\zeta^{d\cdot n/d}-1)}{\zeta^d - 1} = 0.$$

This establishes that $\sum_{k=1}^{n}\gcd(k,n)\zeta^k = \varphi(n)$.

Solution 2: Let $n = 4004$ and $\zeta = e^{2\pi i/n}$, so that $2\cos(\pi k/2002) = \zeta^k + \zeta^{-k}$.

Claim 1. If $\omega = e^{2\pi i/n}$, then

$$g(n) := \sum_{\gcd(k,n)=1, 0\le k<n} \omega^k = \mu(n).$$

Proof. The function

$$f(n) = \sum_{k=0}^{n} e^{2\pi i k/n}$$

is 0 when $n > 1$, and 1 when $n = 1$. On the other hand, by considering $\gcd(k,n)$ (e.g. primitive dth roots of unity for $d \mid n$) we see

$$f(n) = \sum_{d|n} g(d).$$

But this readily implies $g(n) = \mu(n)$ by mobius inversion, as desired. □

We can now calculate:

$$\begin{aligned}
\sum_{k=1}^{n} \gcd(k,n)\frac{\zeta^k+\zeta^{-k}}{2} &= \left(\sum_{k=1}^{n} \gcd(k,n)\frac{\zeta^k}{2}\right) + \left(\sum_{k=1}^{n} \gcd(k,n)\frac{\zeta^{-k}}{2}\right) \\
&= \left(\sum_{k=1}^{n} \gcd(k,n)\frac{\zeta^k}{2}\right) + \left(\sum_{k'=0}^{n} \gcd(n-k',n-1)\frac{\zeta^{n-k'}}{2}\right) \\
&= \sum_{k=1}^{n} \gcd(k,n)\zeta^k \\
&= \sum_{d|n} d \sum_{\gcd(k,n)=d} \zeta^k \\
&= \sum_{d|n} d \sum_{\gcd(j,n/d)=1} (\zeta^d)^j \\
&= \sum_{d|n} d\mu(n/d) \\
&= \varphi(n).
\end{aligned}$$

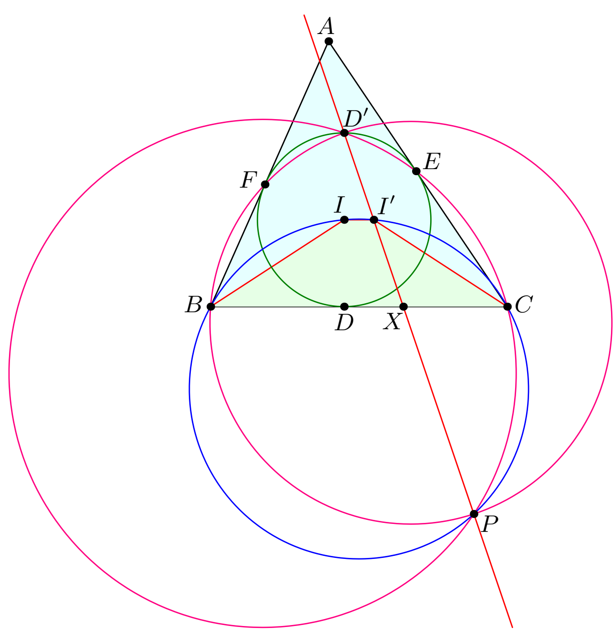
Plugging in $n = 4004$ gives $\varphi(4004) = \boxed{1440}$.

30. **[16]** Let ABC be a triangle with $AB = 60$, $AC = 67$, and $BC = 69$. The incircle ω of triangle ABC touches sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F, respectively. Let D' be the point diametrically opposite to D in ω. Let the common chord of the circumcircles of triangles $BD'F$ and $CD'E$ meet line BC at X. Compute BX.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{45}$

Solution:



The crux of the problem lies in the following claim.

Claim 1. If I' is the reflection of I across perpendicular bisector of BC, then I' lies on the radical axis.

Proof. Let P be second intersection of $\odot(BD'F)$ and $\odot(CD'E)$. Note that

$$\angle BPC = \angle BPD' + \angle CPD' = \angle AFD' + \angle AED' = 90^\circ - \frac{\angle A}{2},$$

so $P \in \odot(BIC)$. Finally, we note that

$$\angle BPI' = \angle BCI' = \frac{\angle B}{2} = \angle AFD' = \angle BPD',$$

so I', D', P are collinear. $\square$

Once we have this, the answer extraction is strikingly clean. We easily compute $s = 98$, so $BD = 31$ and $CD = 38$. If D_1 is the foot from I' to BC, then $BD_1 = 38$, so $DD_1 = 7$. Finally, D_1 is the midpoint of DX, so $XD_1 = 7$, so $BX = BD_1 + D_1X = 38 + 7 = \boxed{45}$.

31. **[16]** Let $\zeta = \cos\frac{2\pi}{19} + i\sin\frac{2\pi}{19}$. It is given that the polynomial $x^3 + 16x^2 + 3x - 229$ has three distinct real roots, and its largest root can be uniquely written in the form

$$a_1\zeta + a_2\zeta^2 + \cdots + a_{18}\zeta^{18}$$

for some rational numbers $a_1, \ldots, a_{18}$. Compute $a_1^2 + a_2^2 + \cdots + a_{18}^2$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{564}$

Solution: Let the three roots be p, q, r, and the root in question is p. The following claim recovers the remaining two roots.

Claim 1. For any $t \in \mathbb{F}_{19}^\times$, the expression

$$R_t = \sum_{j=1}^{18} a_j \zeta^{tj}$$

is still a root of the same cubic. Furthermore, as t ranges across $\mathbb{F}_{19}^\times$, each root of the cubic appears in R_t exactly 6 times.

Proof. Consider the polynomial

$$f(x) = \sum_{j=1}^{18} a_j x^j, \quad g(x) = f(x)^3 + 16f(x)^2 + 3f(x) - 229.$$

Then we have that $f(x)$ (and hence $g(x)$) has integer coefficients. Moreover, $g(\zeta) = 0$. Since the minimal polynomial of ζ is $x^{18} + \cdots + x^2 + x + 1$, it follows that $x^{18} + \cdots + x^2 + x + 1$ divides $g(x)$, which means that $g(\zeta^t) = 0$.

To show that each root appear 6 times, we note that the polynomial

$$\prod_{t \in \mathbb{F}_{19}^\times} (x - R_t)$$

has integer coefficients by the theory of symmetric polynomial. However, since the cubic $x^3 + 16x^2 + 3x - 229$ is irreducible (one only needs to check that ± 1 or ± 229 are not a root), this polynomial must be a power of $x^3 + 16x^2 + 3x - 229$, so we have that $\prod_{t \in \mathbb{F}_{19}^\times}(x - R_t) = (x^3 + 16x^2 + 3x - 229)^6$, implying that each root appears exactly 6 times. $\square$

Thus, summing R_t across all $t \in \mathbb{F}_{19}^\times$ gives

$$-(a_1 + \cdots + a_{18}) = 6(p+q+r) \implies a_1 + \cdots + a_{18} = 96.$$

Next, observe that for any t, we have

$$R_t^2 = |R_t|^2 = \sum_{1 \le j,k \le 18} a_j a_k \zeta^{t(j-k)}$$

Summing this across all $t \in \mathbb{F}_{19}^\times$ gives

$$\begin{aligned} 6(p^2+q^2+r^2) &= 18(a_1^2 + \cdots + a_{18}^2) - 2 \sum_{1 \le j < k \le 18} a_j a_k \\ &= 19(a_1^2 + \cdots + a_{18}^2) - (a_1 + \cdots + a_{18})^2. \end{aligned}$$

By Vieta, $p^2 + q^2 + r^2 = 16^2 - 2 \cdot 3 = 250$, which implies that

$$a_1^2 + \cdots + a_{18}^2 = \frac{6 \cdot 250 + 96^2}{19} = \boxed{564}.$$

Remark. If

$$\begin{aligned} A &= \zeta^1 + \zeta^7 + \zeta^8 + \zeta^{11} + \zeta^{12} + \zeta^{18} \\ B &= \zeta^2 + \zeta^{14} + \zeta^{16} + \zeta^3 + \zeta^5 + \zeta^{17} \\ C &= \zeta^4 + \zeta^9 + \zeta^{13} + \zeta^6 + \zeta^{10} + \zeta^{15}, \end{aligned}$$

then it is straightforward to check that $3A+6B+7C$, $3B+6C+7A$, and $3C+6A+7B$ are the roots of this cubic. Note also that A, B, and C are clearly real. It is interesting to observe that the exponents in A are the cubic residues modulo 19.

32. **[16]** Kelvin the frog starts at the center of a regular hexagon $ABCDEF$ with side length 100, facing towards A. He hops forward an integer distance between 0 and 200 units, inclusive, then turns 120° clockwise. He repeats this process two more times (possibly jumping different distances), ending up within hexagon $ABCDEF$ (possibly on its boundary). Compute the number of distinct paths he could have taken.

Proposed by: Derek Liu

Answer: $\boxed{101^4 - 100^4 = 4060401}$

Solution: Kelvin the frog essentially jumps along a unit triangular grid; any point within 100 steps of the origin along this grid could be his ending spot.

We can view the triangular grid as the 3-dimensional lattice points projected onto the plane $x+y+z=0$ (and scaled accordingly). The hexagon is then the projection of the region satisfying $|x-y| \le 100$, $|y-z| \le 100$, and $|z-x| \le 100$ (in other words, the region in which no two coordinates differ by more than 100).

If Kelvin's jumps have lengths a, b, and c in that order, then his ending position will be the projection of (a,b,c). To count the number of such points, we casework on $\max(a,b,c)$.

- If $\max(a,b,c) \le 100$, then there are 101^3 possible points, as any point in the region $[0,100]^3$ has no two coordinates differ by more than 100.
- If $\max(a,b,c) = m > 100$, then each of a, b, and c must be at least $m-100$. Since their maximum must still be exactly m, the points that satisfy this are all points in $[m-100, m]^3$ minus all points in $[m-100, m-1]^3$. There are 100 possible values for m (101 through 200), so there are $100(101^3 - 100^3)$ points in this case.

The answer is thus

$$101^3 + 100(101^3 - 100^3) = 101^4 - 100^4 = \boxed{4060401}.$$

33. **[20]** From each of the first 6 sets of problems in this Guts round, Mark selects one of the four (correct) answers from that set. Compute the minimum possible range of these 6 values.

Submit a real number E. If the correct answer is A, you will receive $\max(0, \lfloor 20.99 - 18|E - A|^{1/3} \rfloor)$ points.

(The range of n values $x_1 \le x_2 \le \cdots \le x_n$ is given by $x_n - x_1$.)

Proposed by: Derek Liu

Answer: $\boxed{\frac{2251}{144} \approx 15.6319444444444}$

Solution: One set of answers that achieves this range is $(4, 11, 1, 16, 53/144, 6\sqrt{2})$ from problems 3, 7, 10, 15, 20, and 23, respectively.

The range is entirely determined by sets 4 and 5. The correct answers from those sets are in the following table.

	Set 4	Set 5
First problem	$18 \cdot 7^{2024}$	$1 - \frac{\pi}{4} \approx 0.215$
Second problem	136	$\frac{165}{2} = 82.5$
Third problem	16	6723
Fourth problem	$3\sqrt{30} \approx 16.432$	$\frac{53}{144} \approx 0.368$

The minimal positive difference between any set 4 answer and any set 5 answer is $16 - \frac{53}{144} = \boxed{\frac{2251}{144}}$.

34. **[20]** A positive integer is called *good* if it has no prime factors larger than 10^4. Sarunyu picks two odd positive integers a and b, both between 1 and 10^4 (inclusive), independently and uniformly at random. Estimate the expected number of good divisors of $a^2 + b^2$.

Submit a real number E. If the correct answer is A, you will receive $\lfloor 20.05e^{-400(1-E/A)^2} \rfloor$ points.

Proposed by: Rohan Das

Answer: $\boxed{13.12557328}$

Solution: Recall that the number of good divisors of $a^2 + b^2$ is comptued by

$$\tau_{\text{good}}(a^2 + b^2) = \prod_{p \le 10^4} (\nu_p(a^2 + b^2) + 1).$$

In what follows, any product indexed by p is understood to run across only prime numbers. We estimate each random variable $\nu_p(a^2 + b^2)$ as independent. In particular, we estimate $\mathbb{E}[\tau_{\text{good}}(a^2 + b^2)]$ as

$$T = \prod_{p \le 10^4} \Big(\mathbb{E}[\nu_p(a^2 + b^2)] + 1 \Big).$$

We estimate T by separating into cases of p.

- If $p = 2$, then since a and b are always odd, we get that $a^2 + b^2$ is always $2 \pmod 4$, so $\nu_2(a^2 + b^2) = 1$ always, implying that
$$\mathbb{E}[\nu_2(a^2 + b^2)] + 1 = 2.$$

- If $p \equiv 3 \pmod 4$, we claim that $p^k \mid a^2+b^2$ if and only if $p^{\lceil k/2 \rceil}$ divides both a and b. In particular, $\nu_p(a^2+b^2)$ is always even. Proof TBD. Thus, we get that

$$\begin{aligned}\mathbb{E}[\nu_p(a^2+b^2)]+1 &= 1+2\sum_{k\geq 1}\mathbb{P}[\nu_p(a^2+b^2)\geq 2k] \\ &\approx 1+2\sum_{k\geq 1}\frac{1}{p^{2k}} \\ &= 1+\frac{2}{p^2-1}=\frac{p^2+1}{p^2-1},\end{aligned}$$

contributing $\frac{p^2+1}{p^2-1}$ into the product.

- If $p \equiv 1 \pmod 4$, then first count the number of (a,b) such that $1 \leq a, b \leq p^k$ and $p^k \mid a^2+b^2$. Let this number be N_k. Then we have two cases.
  - If p divides both a and b, then let $a' = \frac{a}{p}$ and $b' = \frac{b}{p}$. Then we need $1 \leq a', b' \leq p^{k-1}$ and $p^{k-2} \mid a^2+b^2$, so there are $p^2 N_{k-2}$ solution.
  - If p does not divide either a or b, then by a routine application of Hensel's lemma, fixing any b not divisible by p gives exactly two solutions for a. Thus, there are $2p^{k-1}(p-1)$ solutions.

  This establishes the recurrence relation

$$N_k = p^2 N_{k-2} + 2p^{k-1}(p-1).$$

  By a straightforward induction, with $N_0 = 1$ and $N_1 = 2p-1$, we get that

$$N_k = (k+1)p^k - kp^{k-1}.$$

  Thus, we get that

$$\mathbb{P}(\nu_p(a^2+b^2)\geq k)\approx \frac{N_k}{p^{2k}} = \frac{k+1}{p^k}-\frac{k}{p^{k+1}},$$

  so we can approximate

$$\begin{aligned}\mathbb{E}[\nu_p(a^2+b^2)+1] &= 1+\sum_{k=1}^{\infty}\mathbb{P}(\nu_p(a^2+b^2)\geq k) \\ &\approx 1+\sum_{k=1}^{\infty}\frac{k+1}{p^k}-\frac{k}{p^{k+1}} \\ &= 1+\left(\frac{2}{p}-\frac{1}{p^2}\right)+\left(\frac{3}{p^2}-\frac{2}{p^3}\right)+\left(\frac{4}{p^3}-\frac{3}{p^4}\right)+\dots \\ &= 1+\frac{2}{p}+\frac{2}{p^2}+\frac{2}{p^3}+\dots \\ &= 1+\frac{2}{p-1}=\frac{p+1}{p-1},\end{aligned}$$

  contributing $\frac{p+1}{p-1}$ to the product.

Therefore, we have

$$T \approx 2\cdot \prod_{\substack{p\equiv 1 \bmod 4 \\ p\leq 10^4}}\frac{p+1}{p-1}\cdot \prod_{\substack{p\equiv 3 \bmod 4 \\ p\leq 10^4}}\frac{p^2+1}{p^2-1}.$$

To compute this, we estimate each term separately.

First, we will attempt to estimate

$$\prod_{p\leq 10^4}\frac{p+1}{p-1}$$

over all primes $p \leq 10^4$. Let $N = 10^4$. We rewrite this product as

$$\prod_{p \leq N} \frac{(p^2-1)/p^2}{(p-1)^2/p^2} = \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-2} \cdot \prod_{p \leq N} \left(1 - \frac{1}{p^2}\right).$$

Note that

$$\prod_{p} \left(1 - \frac{1}{p^2}\right)^{-1} = \prod_{p} \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \dots\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Since this product is convergent, restricting to $p \leq N$ won't change the product meaningfully when N is large. To evaluate the other term, we use Mertens' theorem, which states

$$\prod_{p \leq N} \left(1 - \frac{1}{p}\right) \sim e^{\gamma} \ln N,$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Using the Taylor series expansion of e^x, we can approximate

$$e^{\gamma} \approx 1 + \gamma + \frac{\gamma^2}{2} + \frac{\gamma^3}{6} \approx 1.78.$$

Also,

$$\ln 10000 = 4 \ln 10 \approx 4 \cdot 2.3 = 9.2,$$

so

$$\prod_{p \leq N} \frac{p+1}{p-1} \approx \frac{(1.78 \cdot 9.2)^2}{\pi^2/6} \approx 163.$$

We can approximate the product restricted to $p \equiv 1 \bmod 4$ by noting that large primes tend to be evenly split between $1 \bmod 4$ and $3 \bmod 4$, though since the smallest primes contribute the largest terms, we should consider them separately:

$$\begin{aligned}
\prod_{\substack{p \equiv 1 \bmod 4 \\ p \leq 10^4}} \frac{p+1}{p-1} &\approx \frac{5+1}{5-1} \cdot \sqrt{\prod_{13 \leq p \leq N} \frac{p+1}{p-1}} \\
&\approx \frac{6}{4} \cdot \sqrt{163 \cdot \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{4}{6} \cdot \frac{6}{8} \cdot \frac{10}{12}} \\
&= \frac{6}{4} \cdot \frac{1}{6} \sqrt{407.5} \\
&\approx \frac{1}{4} \left(20 + \frac{7.5}{40}\right) \\
&\approx 5.047.
\end{aligned}$$

As for the other term, note that

$$\prod_{p} \frac{p^2+1}{p^2-1} \leq \prod_{p} \left(1 - \frac{1}{p^2}\right)^{-2} = \left(\frac{\pi^2}{6}\right)^2$$

converges, so any product of such terms should be completely dominated by the first several terms. Restricting to $p \equiv 3 \bmod 4$, the first two terms are $\frac{10}{8}$ and $\frac{50}{48}$, giving us a final estimate of

$$2 \cdot 5.047 \cdot \frac{10}{8} \cdot \frac{50}{48} \approx 13.143,$$

an estimate close enough for all 20 points, though we note that this estimate makes many rough approximations and its accuracy is not entirely indicative of its soundness.

35. **[20]** Estimate

$$\log_{10}\left(\sum_{k=0}^{15000}\frac{30000!}{(k!)^2(30000-2k)!}\right).$$

Submit a real number E. If the correct answer is A, you will receive $\lfloor 20.05e^{-0.69(E-A)^2}\rfloor$ points.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{14311.088033944017}$

Solution: When 10000 is replaced by n, we claim that the exact asymptotic formula is

$$\sum_{k=0}^{3n/2}\binom{3n}{k,k,3n-2k}\sim\frac{3^{3n}}{2\sqrt{\pi n}}.$$

(Recall the standard notation that $f(n)\sim g(n)$ if and only if $\lim_{n\to\infty}f(n)/g(n)=1$.)

Taking $\log_{10}$ of the right hand side gives a very accurate estimate of

$$30000\log_{10}3-2-\log_{10}\left(2\sqrt{\pi}\right)=14311.0880367,$$

which is off by $2.8\cdot 10^{-6}$, more than enough to get 20 points.

Before we prove the asymptotic formula, we show how to derive it heuristically. Let $f(k)=\binom{3n}{k,k,3n-2k}$. It is straightforward to show that $f(k)$ is maximized at $k=n$ (though we won't use this in the actual estimation). Thus, we try to estimate the value of $f(k)$ relative to $f(n)$. To that end, we note that if $x\ll n$, then

$$\frac{f(n+x)}{f(n+x-1)}=\frac{(n-2x+2)(n-2x+1)}{(n+x)^2}\approx\frac{\left(1-\frac{2x}{n}\right)^2}{\left(1+\frac{x}{n}\right)^2}=1-\frac{6x}{n}\approx e^{-6x/n}.$$

Thus, we have that

$$\frac{f(n+x)}{f(n)}=\exp\left(-\frac{6}{n}(1+2+\cdots+x)\right)\approx\exp(-3x^2/n),$$

so we have that the sum is approximately

$$\sum_{k=0}^{3n/2}f(k)\approx\sum_{x=-\infty}^{\infty}\exp(-3x^2/n)f(n)\approx\int_{-\infty}^{\infty}\exp(-3x^2/n)f(n)\,dx=\sqrt{\frac{\pi n}{3}}f(n),$$

where the integral can be easily evaluated using Gaussian integral $\int_{-\infty}^{\infty}e^{-x^2}\,dx=\sqrt{\pi}$. Finally, since $f(n)=\frac{(3n)!}{n!^3}$, we use Stirling's formula $n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ to estimate $f(n)$, getting that the sum is

$$\sum_{k=0}^{3n/2}f(k)\sim\sqrt{\frac{\pi n}{3}}f(n)\sim\sqrt{\frac{\pi n}{3}}\cdot\frac{(2\pi\cdot 3n)^{1/2}\left(\frac{3n}{e}\right)^{3n}}{(2\pi n)^{3/2}\left(\frac{n}{e}\right)^{3n}}\sim\frac{3^{3n}}{2\sqrt{\pi n}}.$$

To prove this rigorously, we need to keep track of error terms. If $x\le n$, then we have the estimate $e^{x/n}=1+\frac{x}{n}+O\left(\frac{x^2}{n^2}\right)$, with the constant in the error term not depending on n or x. Using this, we get that

$$\begin{aligned}\frac{f(n+x)}{f(n+x-1)}&=\frac{(n-2x+2)(n-2x+1)}{(n+x)^2}\\&=\frac{\left(1-\frac{2x}{n}+O\left(\frac{x^2}{n^2}\right)\right)^2}{\left(1+\frac{x}{n}+O\left(\frac{x^2}{n^2}\right)\right)^2}\\&=1-\frac{6x}{n}+O\left(\frac{x^2}{n^2}\right)\\&=\exp\left(-\frac{6x}{n}+O\left(\frac{x^2}{n^2}\right)\right).\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}\frac{f(n+x)}{f(n)} &= \exp\left(-\frac{6}{n}(1+2+\cdots+x)+O\left(\frac{1^2+\cdots+x^2}{n^2}\right)\right)\\ &= \exp\left(-\frac{3x^2}{n}+O\left(\frac{x^3}{n^2}\right)\right).\end{aligned}$$

We now estimate the sum $\sum_{k=0}^{3n/2} f(k)$ by splitting into two parts.

- **If $|k-n|>n^{0.6}$**, then letting $x=k-n$, then we have that the error term $-\frac{x^3}{n^2}$ is much smaller than the main term $-\frac{3x^2}{n}$. In particular, we get that $f(n+x)\le \exp\left(-n^{0.2}\right)f(n)$, so we get that

$$\sum_{|k-n|>n^{0.6}} f(k)\le n\exp(-n^{0.2})f(n)\ll f(n),$$

  so this part of the sum is dominated by the maximum term $f(n)$.
- **If $|k-n|<n^{0.6}$**, then if $x=k-n$, then $f(k)/f(n)=(1+o(1))\exp(-3x^2/n)$. Therefore, we have that

$$\sum_{|k-n|<n^{0.6}} f(k)\sim f(n)\sum_{x=-n^{0.6}}^{n^{0.6}}\exp\left(-\frac{3x^2}{n}\right).$$

  To estimate this sum, we can bound it by an integral. Since the function $g(x)=e^{-3x^2/n}$ is monotone on $x\le 0$ and $x\ge 0$, and $g(0)=1$, we get that

$$\sum_{x=-n^{0.6}}^{n^{0.6}}\exp\left(-\frac{3x^2}{n}\right)=\int_{-n^{0.6}}^{n^{0.6}}\exp\left(-\frac{3x^2}{n}\right)dx+O(1).$$

  Finally, we estimate the integral by extending it to infinity and estimating its tail. We get that

$$\begin{aligned}\sum_{x=-n^{0.6}}^{n^{0.6}}\exp\left(-\frac{3x^2}{n}\right) &= \int_{-n^{0.6}}^{n^{0.6}}\exp\left(-\frac{3x^2}{n}\right)dx+O(1).\\ &= \int_{-\infty}^{\infty}\exp\left(-\frac{3x^2}{n}\right)dx+2\int_{n^{0.6}}^{\infty}\exp\left(-\frac{3x^2}{n}\right)dx+O(1)\\ &= \sqrt{\frac{\pi n}{3}}+O\left(\int_{n^{0.6}}^{\infty}x\exp\left(-\frac{3x^2}{n}\right)dx\right)+O(1)\\ &= \sqrt{\frac{\pi n}{3}}+O\left(\frac{n}{6}\exp\left(-\frac{3\cdot n^{2\cdot 0.6}}{n}\right)\right)+O(1)\\ &= \sqrt{\frac{\pi n}{3}}+O(1).\end{aligned}$$

  Hence, we have that

$$\sum_{|k-n|<n^{0.6}} f(k)\sim\left(\sqrt{\frac{\pi n}{3}}+O(1)\right)f(n).$$

Thus, by combining two cases, we have that

$$\sum_{k=0}^{3n/2} f(k)\sim\sqrt{\frac{\pi n}{3}}f(n).$$

Finally, since $f(n)=\frac{(3n)!}{n!^3}$, we can use Stirling's formula as above to get the claimed asymptotic formula.

Remark. This sum is the number of possible moves that Marin can do in Team Round Problem 8 when $n = 30000$.

36. **[20]** Sebastian is going for a walk in the coordinate plane. He starts at the origin facing in the positive x direction. Each minute, he takes a step forward, then randomly chooses one of the three axial directions other than the opposite of his current orientation. Sebastian stops walking once he returns to a point he has already visited. Estimate the expected number of steps Sebastian walks.

Submit a real number E. If the correct answer is A, you will receive $\lfloor 20.05e^{-0.3(E-A)^2} \rfloor$ points.

Proposed by: Jacob Paltrowitz

Answer: $\boxed{\approx 14.0844}$

Solution: The simplest way for Sebastian to stop walking is for him to walk in a 1×1 square. At any time step after three steps, the probability that his last 3 steps cause him to walk in a square (and thus necessarily self-intersect) is $\frac{2}{27}$. So, we can crudely upper bound the length of the walk by assuming that this is the only reason why Sebastian will ever stop. If we assume that the survival of the random walk is independent of previous steps, we can approximate the probability of lasting exactly n steps as

$$\left(\frac{25}{27}\right)^{n-4} \left(\frac{2}{27}\right).$$

This gives a corresponding expected value of

$$\sum_{n=4}^{\infty} n \left(\frac{25}{27}\right)^{n-4} \left(\frac{2}{27}\right) = 16.5$$

which is good for 3 points.

To improve this bound, we consider the next smallest way for Sebastian to self intersect. This would involve him walking along the edge of a 2×1 rectangle. At any given time $n > 6$, this occurs with probability $\frac{6}{243} = \frac{2}{81}$. So, we can update our probability estimate of lasting exactly n (for $n > 6$) to be

$$\left(\frac{25}{27}\right)^{n-4} \left(\frac{79}{81}\right)^{n-6} \left(\frac{2}{27} + \frac{2}{81}\right).$$

Plugging back in, this gives an expected value of

$$\frac{2}{27}(4) + \frac{25}{27} \cdot \frac{2}{27}(5) + \sum_{n=6}^{\infty} n \left(\frac{25}{27}\right)^{(n-4)} \left(\frac{79}{81}\right)^{(n-6)} \left(\frac{2}{27} + \frac{2}{81}\right) \approx 14.018$$

which is enough for 20 points.