HMMT February 2026

February 14, 2026

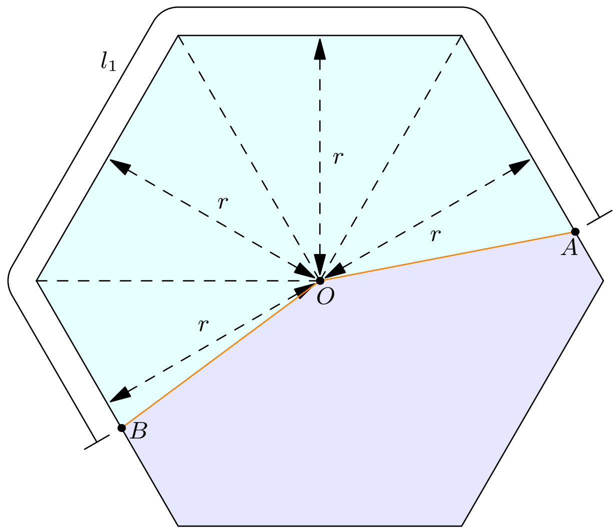
Team Round

1. **[20]** Let $\mathcal{P}$ be a regular hexagon with center O. Distinct points A and B lie on the boundary of $\mathcal{P}$. The segments OA and OB divide the interior of $\mathcal{P}$ into two polygons $\mathcal{R}_1$ and $\mathcal{R}_2$. Let a_1 and a_2 be the areas of $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively, and let p_1 and p_2 be the perimeters of $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. Given that $a_1/p_1 = a_2/p_2$, prove that $a_1 = a_2$.

   Proposed by: Linus Yifeng Tang

   Answer: N/A

   Solution:



Let r be the inradius of the polygon. Let

- l_1 be the length of the shared boundary of $\mathcal{R}_1$ and $\mathcal{P}$,
- l_2 be the length of the shared boundary of $\mathcal{R}_2$ and $\mathcal{P}$, and
- k be the length of the shared boundary of $\mathcal{R}_1$ and $\mathcal{R}_2$, i.e., $PA + PB$.

Then the area of $\mathcal{R}_1$ is $a_1 = \frac{rl_1}{2}$, as we can divide $\mathcal{R}_1$ into triangles each of height r with total base length l_1. Likewise, the area of $\mathcal{R}_2$ is $a_2 = \frac{rl_2}{2}$. Meanwhile, the perimeters of $\mathcal{R}_1$ and $\mathcal{R}_2$ are $p_1 = l_1 + k$ and $p_2 = l_2 + k$, respectively.

The condition $a_1/p_1 = a_2/p_2$ thus implies that

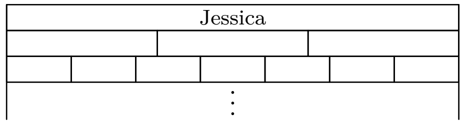
$$\frac{rl_1/2}{l_1 + k} = \frac{rl_2/2}{l_2 + k}.$$

Dividing both sides by $r/2$ and cross-multiplying, we get

$$l_1(l_2 + k) = l_2(l_1 + k),$$

which implies that $l_1 k = l_2 k$, so $l_1 = l_2$. Hence, $a_1 = rl_1/2 = rl_2/2 = a_2$, as desired.

2. **[25]** Jessica the jackrabbit wants to climb down a wall. The wall consists of 2026 horizontal layers stacked vertically. The nth layer from the top is partitioned into $2^n - 1$ identical rectangular bricks arranged side by side. Jessica begins in the topmost layer, which contains a single brick. A move consists of Jessica going down one layer to a brick that shares a side with the brick she is currently on.



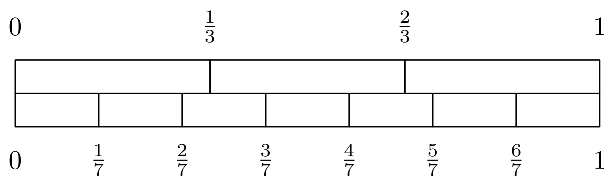
Determine, with proof, the total number of distinct sequences of moves Jessica can take to reach the 2026th layer.

Proposed by: Isabella Zhu

Answer: $\boxed{3^{2025}}$

Solution 1: The answer is $\boxed{3^{2025}}$. The key observation is that each brick on layer n shares a side with exactly 3 bricks on layer $n+1$, so Jessica always has 3 ways to move down one layer. Since she moves down 2025 layers, she can make 3^{2025} sequences of moves.

To prove this, we directly show case-by-case that each brick shares a side with exactly 3 bricks below. We label the position of the block as the following: fix n, where $1 \le n \le 2025$, and let $N = 2^n - 1$, so that $2^{n+1} - 1 = 2N + 1$. Let the left and right edges of the wall be at $x = 0$ and $x = 1$ respectively. Below is a diagram for $n = 2$.



Claim 1. We have $\frac{2a}{2N+1} < \frac{a}{N} < \frac{2a+1}{2N+1}$ for all $1 \le a \le N - 1$.

Proof. For the lower bound, write

$$\frac{2a}{2N+1} < \frac{2a}{2N} = \frac{a}{N}.$$

For the upper bound, note that

$$\frac{2a+1}{2N+1} > \frac{a}{N} \iff 1 - \frac{2a+1}{2N+1} < 1 - \frac{a}{N} \iff \frac{2(N-a)}{2N+1} < \frac{N-a}{N}.$$

The last inequality holds because

$$\frac{2(N-a)}{2N+1} < \frac{2(N-a)}{2N} = \frac{N-a}{N}.$$

□

Claim 2. For all $1 \le n \le 2025$, each brick on layer n shares a side with exactly 3 bricks on layer $n+1$.

Proof. Consider the a^{th} brick on layer n borders exactly 3 bricks on layer $n+1$, namely bricks $2a-1$, $2a$, and $2a+1$. We split into cases based on a:

- If $a = 1$, then the brick's left edge is at 0 and its right edge is at $\frac{1}{N}$. By the claim, $\frac{1}{N}$ is strictly between $\frac{2}{2N+1}$ and $\frac{3}{2N+1}$, which are the edges of brick 3 on layer $n+1$. Hence brick 1 on layer n borders bricks 1, 2, and 3 on layer $n+1$.
- If $a = N$, then the brick's right edge is at 1 and its left edge is at $\frac{N-1}{N}$. By the claim, $\frac{N-1}{N}$ is strictly between $\frac{2N-2}{2N+1}$ and $\frac{2N-1}{2N+1}$, which are the edges of brick $2N-1$ on layer $n+1$. Hence brick N on layer n borders bricks $2N-1$, $2N$, and $2N+1$ on layer $n+1$.
- If $1 < a < N$, then:
  - The brick's left edge is at $\frac{a-1}{N}$, which is strictly between $\frac{2a-2}{2N+1}$ and $\frac{2a-1}{2N+1}$. These are the edges of brick $2a-1$ on layer $n+1$.

– The brick's right edge is at $\frac{a}{N}$, which is strictly between $\frac{2a}{2N+1}$ and $\frac{2a+1}{2N+1}$. These are the edges of brick $2a+1$ on layer $n+1$.

Hence brick a on layer n borders bricks $2a-1$, $2a$, and $2a+1$ on layer $n+1$.

With that, we've shown that every brick on layer n shares a side with exactly 3 bricks on layer $n+1$, as desired. $\square$

This claim implies Jessica always has 3 ways to move down one layer. Since she moves down 2025 layers, she has 3^{2025} possible sequences of moves.

Solution 2: We present an alternative proof that each brick on layer n shares a side with exactly 3 bicks on layer $n+1$. Instead of working case-by-case, we show that, for each brick in layer n, the number of bricks on layer $n+1$ sharing sides with that brick cannot be at most 2 nor at least 4.

Similar to the first solution, fix n, where $1 \le n \le 2025$, and let $N = 2^n - 1$. Let the left and right edges of the wall be at $x = 0$ and $x = N(2N+1)$, respectively.

In this setup, the vertical edges of the bricks on layer n are at x-coordinates that are multiples of $2N+1$, and the edges of the bricks on layer $n+1$ are at x-coordinates that are multiples of N. In particular, all edges of bricks are at integer x-coordinates. Below is a diagram for $n = 2$:

Consider any brick B on layer n, with left and right edges at A and $A + 2N + 1$, respectively. We'll prove B borders exactly 3 bricks on layer $n+1$.

- Suppose that B borders at most 2 bricks on layer $n+1$. The bottom edge of B has length $2N+1$. The bricks on layer $n+1$ have length N, so they cover a length of at most $2N$ on this edge, contradiction.
- Suppose that B borders at least 4 bricks on layer $n+1$. Let E_1 be the vertical border between the first and second (from the left) of these bricks. E_1 must have x-position strictly greater than A. Since all borders between bricks are at integer coordinates, E_1's x-position is at least $A+1$. Similarly, let E_2 be the vertical border between the last and second-to-last of these bricks; E_2 must have x-position at most $A + 2N$.

  By assumption, there must be at least 2 bricks between E_1 and E_2. These bricks fit inside an interval of length $(A+2N) - (A+1) = 2N - 1$. But this is a contradiction because 2 bricks have total length $2N$.

We've reached a contradiction in both cases, so we're done.

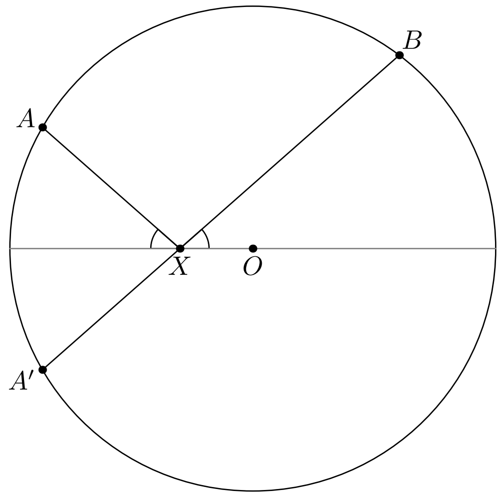
3. **[30]** Let α and β be complex numbers such that $\alpha\beta + \alpha + \beta + 100 = 0$. Suppose that $|\alpha| = |\beta| = M$ for some nonnegative real number M. Determine, with proof, all possible values of M.

Proposed by: Jason Mao

Answer: $\boxed{10}$

Solution 1: The answer is $M = 10$, achievable by $(\alpha, \beta) = (-10, 10)$, for example. It remains to show that this is the only possible value.

Let A and B be the points corresponding to α and β in the complex plane. The condition $|\alpha| = |\beta| = M$ implies that A and B lie on the circle ω centered at the origin O.



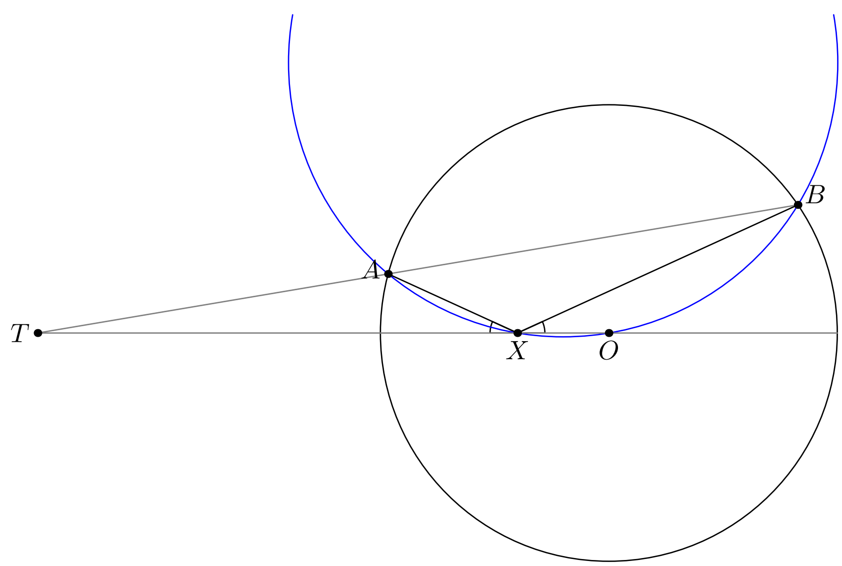
To translate the condition $\alpha\beta+\alpha+\beta+100$, we let X be the point -1. Then we rewrite that condition to

$$(\alpha+1)(\beta+1)=-99.$$

In particular, the arguments of $\alpha+1$ and $\beta+1$ sum to $180°$. Therefore, AX and BX make the same angle to the x-axis. Moreover, since the right hand side -99 is negative, we get that A and B lies on the same side with respect to OX. In particular, OX externally bisects $\angle AXB$. Furthermore, $AX \cdot BX = 99$ by comparing magnitudes.

Let A' denote the reflection of A across $\overline{OX}$. The angle condition in the previous paragraph implies A', X, B are collinear. Then $A'X \cdot XB = 99$ implies the power of X with respect to ω is -99. So $OX^2 - M^2 = -99$, and since $OX = 1$, it must be that $M = 10$.

Solution 2: As in the first solution, we show that 10 is the only possible value.



We use the geometric interpretation in the previous solution. In particular, let X be point -1, O be the origin, and A and B be points corresponding to α and β, respectively. Then, we have that OX externally bisects $\angle AXB$ and $OA = OB = M$, implying that $AXOB$ is cyclic. Additionally, $AX \cdot BX = 99$.

Let AB intersects OX at point T. Observe that

- Triangles XAT and XOB are similar, so $XT \cdot XO = XA \cdot XB = 99$. (This is known as $\sqrt{XA \cdot XB}$ inversion.) Since $OX = 1$, we get that $XT = 99$.
- Triangles OXA and OTB are similar, so $OX \cdot OT = OA^2 = M^2$. (This is known as "shooting lemma".) However, $OX = 1$ and $OT = 100$, so $M^2 = 100$, implying that $M = 10$.

Solution 3: As in the first solution, we show that 10 is the only possible value.

We rewrite the equation as

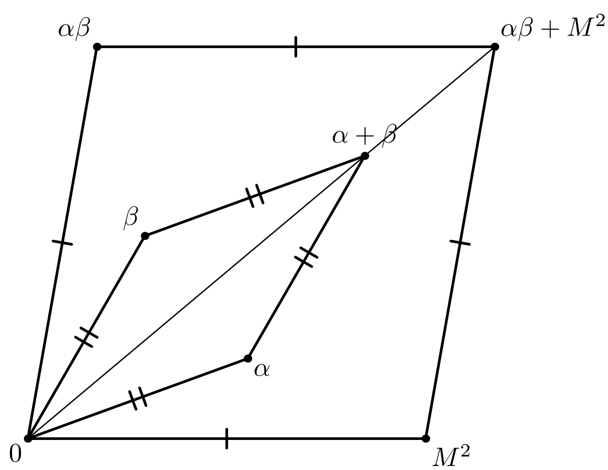
$$(\alpha\beta + M^2) + (\alpha + \beta) = M^2 - 100.$$

The crux of the problem is the following observation.

Claim 1. We have that

$$\frac{\alpha\beta + M^2}{\alpha + \beta} \in \mathbb{R}.$$

Proof 1 (Geometric). Note that since $|\alpha| = |\beta|$, we get that $\alpha + \beta$ lies on the internal angle bisector of angle formed by points α, 0, and β. Similarly, since $|\alpha\beta| = M^2$, we get that $\alpha\beta + M^2$ lies on the internal angle bisector of angle formed by points 1, 0, and $\alpha\beta$.



However, those two angle bisectors are the same, so $\alpha\beta + M^2$ and $\alpha + \beta$ have the same argument. □

Proof 2 (Algebraic). Let $T = \frac{\alpha\beta + M^2}{\alpha + \beta}$. We compare T against the conjugate $\overline{T}$. From $|\alpha| = |\beta| = M$, we get that $\overline{\alpha} = \frac{M^2}{\alpha}$ and $\overline{\beta} = \frac{M^2}{\beta}$, so

$$\overline{T} = \frac{\overline{\alpha}\overline{\beta} + M^2}{\overline{\alpha} + \overline{\beta}} = \frac{\frac{M^4}{\alpha\beta} + M^2}{\frac{M^2}{\alpha} + \frac{M^2}{\beta}} = \frac{\alpha\beta + M^2}{\alpha + \beta} = \overline{T},$$

so $T \in \mathbb{R}$ as desired. □

Since $\alpha\beta + M^2$ and $\alpha + \beta$ are real multiples of each other and they sum to a real number, we have two possibilities: either $M^2 = 100$ or $\alpha + \beta$ is real. The former implies $M = \pm 10$, but $M > 0$, so $M = 10$. Thus, we show that the latter is impossible.

Assume that $\alpha + \beta$ is real. Then $\alpha\beta + M^2$ is real, and so $\alpha\beta$ is real. Thus, we have two cases.

- **If α and β are both real**, then since $|\alpha| = |\beta| = M$, we have that $\alpha, \beta \in \{M, -M\}$. Testing all four possibilities gives $\alpha\beta + \alpha + \beta \in \{-M^2, M^2 - 2M, M^2 + 2M\}$, each of these could not be -100.

- **If α and β are conjugates**, then we have that $\alpha\beta = M^2$, so $\alpha + \beta = -M^2 - 100$. Therefore, $\operatorname{Re}\alpha = -\frac{M^2+100}{2} < -M$, contradicting $|\alpha| = M$.

Solution 4: As in the first solution, we show that 10 is the only possible value.

Clearly $M \neq 0$, so α and β are both nonzero and thus have conjugates given by $\bar{\alpha} = \frac{M^2}{\alpha}$ and $\bar{\beta} = \frac{M^2}{\beta}$. The conjugate of the relationship $\alpha\beta + \alpha + \beta + 100 = 0$ then yields:

$$\begin{aligned} \frac{M^4}{\alpha\beta} + \frac{M^2}{\alpha} + \frac{M^2}{\beta} + 100 &= 0 \\ 100\alpha\beta + M^2(\alpha+\beta) + M^4 &= 0 \\ 100\alpha\beta - M^2(\alpha\beta + 100) + M^4 &= 0 && (\text{since } \alpha+\beta = -\alpha\beta - 100) \\ (100 - M^2)\alpha\beta + (M^4 - 100M^2) &= 0 \\ (100 - M^2)(\alpha\beta - M^2) &= 0 \end{aligned}$$

Thus, either $M^2 - 100 = 0$ or $\alpha\beta - M^2 = 0$. The former implies $M = \pm 10$, but $M > 0$, so $M = 10$. Thus, we show that the latter is impossible.

Suppose that $\alpha\beta = M^2$. Then from $|\alpha| = |\beta| = M$, we know that α and β are conjugates. Since $\alpha + \beta = -(\alpha\beta + 100) = -(M^2 + 100)$, we get that $\operatorname{Re}\alpha = -\frac{M^2+100}{2} < -M$, contradicting $|\alpha| = M$.

4. **[35]** A set of rational numbers S is called *inclusive* if 0 is not an element of S, and for any (not necessarily distinct) elements x, y, and z of S, the number $xy + z$ is also an element of S. Determine, with proof, all rational numbers a for which there exists an inclusive set containing a.

   Proposed by: Henrick Rabinovitz

   Answer: All rational numbers except 0 and $-\frac{1}{n}$ for all positive integers n.

   Solution: We first show that for $a \in \{0\} \cup \{-\frac{1}{n} \mid n \in \mathbb{N}\}$, there is no inclusive set containing a. By definition, there is not an inclusive set containing 0.

   Let $a = -\frac{1}{n}$ for $n \in \mathbb{N}$. Suppose there is an inclusive set S containing a.

   Claim 1. For any integer $k \geq 0$, $\frac{k-n}{n^2} \in S$.

   Proof. We use induction on k. The base case $k = 0$ is true as $a = \frac{-n}{n^2}$. Now assume $\frac{k-n}{n^2} \in S$ for a fixed k. Then
$$\frac{k-n}{n^2} + \left(-\frac{1}{n}\right)\left(-\frac{1}{n}\right) = \frac{k+1-n}{n^2} \in S$$
   which completes the inductive step. □

   Plugging in $k = n$ yields that $0 \in S$, which is a contradiction.

   Now we show that for all other elements $a \in \mathbb{Q}$, there exists an inclusive set containing a. Observe that $\{x \in \mathbb{Q} \mid x > 0\}$ is an inclusive set as if x, y, z are positive rational numbers, $xy + z$ is as well. Therefore it suffices to show that if $a = -\frac{m}{n}$, where $m, n \in \mathbb{N}$, $m > 1$, and $\gcd(m, n) = 1$, then there exists an inclusive set containing a.

   The following claim finishes the problem.

   Claim 2. The set
$$T = \{x \in \mathbb{Q} \mid x \equiv a \pmod{m^2}\}$$
   is an inclusive set containing a.

   Two rational numbers are equivalent modulo an integer n if their difference in simplest terms has numerator divisible by n.

Proof. Clearly $a \in T$. Because $m \neq 1$, $m^2 \nmid m$, so $a \not\equiv 0 \pmod{m^2}$. Therefore $0 \notin T$. Lastly, if x, y, and z are in T, then

$$xy + z \equiv a^2 + a \equiv \left(\frac{m}{n}\right)^2 + a \equiv a \pmod{m^2}$$

so $xy + z \in T$. As we have checked all necessary conditions, the claim is proven. □

5. **[40]** The numbers 1, 2, …, 2026 are written on a blackboard. An operation consists of replacing any number on the blackboard with the positive difference between the largest and smallest numbers currently on the blackboard. Determine, with proof, the least number of operations required to make all the numbers on the blackboard equal.

Proposed by: Srinivas Arun

Answer: $\boxed{3037}$

Solution: Let $2026 = 2n$.

Lemma 1. The final number on the board is at most n.

Proof. Suppose k is the final number. The numbers on the board before the last operation must be $k, k, \ldots, k, 2k$. Since the maximum of the numbers of the board can never increase, we have

$$2k \leq 2n \implies k \leq n.$$

□

Lemma 2. For any $0 \leq k < 2n$, at least $2n - k - 1$ operations are required to make the difference between the largest and smallest numbers equal to k.

Proof. Suppose the difference between the largest and smallest numbers is k. There are at most $k + 1$ distinct numbers on the board, so at least $2n - k - 1$ numbers were originally on the board and are no longer on the board. Hence, at least $2n - k - 1$ operations have been made. □

Lemma 3. At least $3n - 2$ operations are required to make all numbers on the blackboard equal.

Proof. Suppose the final number is k. From the lemmas above, at least $2n - k - 1$ operations are needed before the difference becomes k, and so at most 1 number is k before this point. After this point, at least $2n - 1$ additional operations are required replace the other numbers and finish. Hence, the number of total operations is at least

$$(2n - k - 1) + (2n - 1) = 4n - k - 2 \geq 3n - 2.$$

□

Lemma 4. The upper bound of $3n - 2$ is achievable.

Proof. We can repeatedly replace the smallest number on the board until the numbers are

$$n, n + 1, n + 1, \ldots, 2n - 1, 2n - 1, 2n.$$

Now, replace the remaining numbers with n, replacing $2n$ last. This construction saturates all above inequalities and gives the $3n - 2$ bound. □

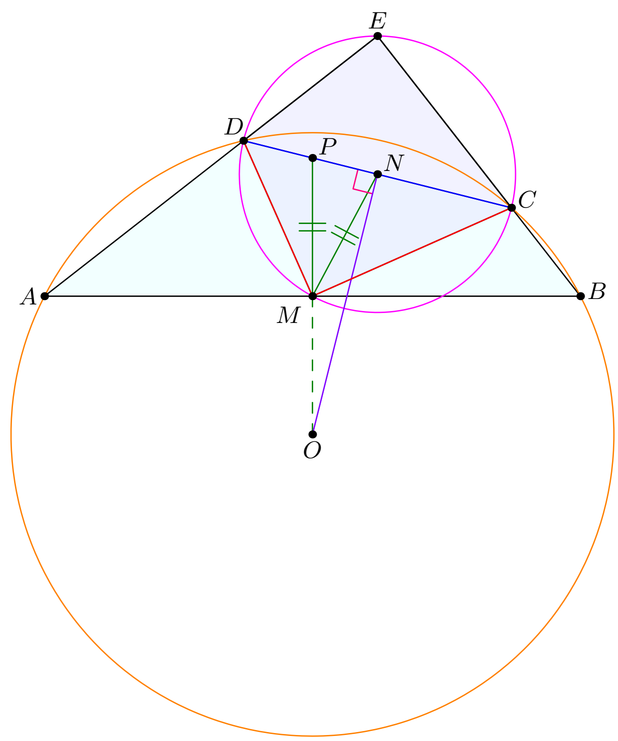
Hence, the answer is $3n - 2 = 3037$.

6. **[40]** Let $ABCD$ be a cyclic quadrilateral with circumcenter O, and let M be the midpoint of $\overline{AB}$. Suppose that $\angle CMD = 90^\circ$ and $\overline{AD} \perp \overline{BC}$. Prove that the reflection of O over $\overline{AB}$ lies on line CD.

Proposed by: Tiger Zhang

Answer:

Solution 1:



Let $\overline{AD}$ and $\overline{BC}$ intersect at E. If E lies on rays DA and CB, then M lies inside the circle with diameter $\overline{CD}$, so $\angle CMD > 90^\circ$, a contradiction. Thus, E lies on rays AD and BC.

Claim 1. Line CD is the perpendicular bisector of $\overline{EM}$.

Proof. Since $\angle CED = \angle CMD = 90^\circ$, quadrilateral $CEDM$ is cyclic. We have

$$\angle CME = \angle CDE = 180^\circ - \angle CDA = \angle ABE = \angle BEM,$$

so $CE = CM$. Analogously, $DE = DM$, so $\triangle CDE \cong \triangle CDM$, implying $\overline{CD}$ is the perpendicular bisector of $\overline{EM}$. □

Let N be the midpoint of $\overline{CD}$; notice that N is the circumcenter of $CEDM$. Let $\overline{CD}$ and $\overline{MO}$ intersect at P. It suffices to show $MO = MP$.

Claim 2. $MN = MP$.

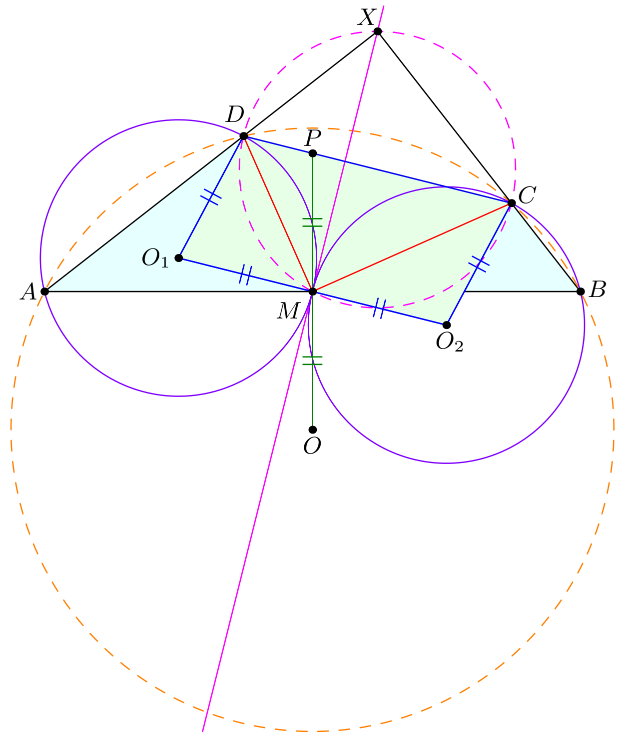
Proof. Assume WLOG that D, N, P, and C are collinear in this order. We have

$$\angle NPM = 90^\circ - \angle EMP = \angle AME = 2\angle BEM = \angle CNM,$$

so $MN = MP$. □

Notice that $\angle ONP = 90^\circ$. Since the midpoint of $\overline{OP}$ is the unique point that lies on both $\overline{OP}$ and the perpendicular bisector of $\overline{NP}$, we conclude M is this midpoint, so $MO = MP$, as desired.

Solution 2:



Let E be the intersection of AD and BC. Similar to the first solution, E lies on ray AD and BC. Let ω_1, ω_2 be the circumcircles of $\triangle ADM$, $\triangle BCM$ with centers O_1, O_2 respectively.

Claim 3. ω_1 and ω_2 are symmetric with respect to point M, and both are tangent to EM.

Proof. Because $MDEC$ and $ABCD$ are cyclic, we have $\angle EMC = \angle EDC = \angle EBA$ and $\angle EMD = \angle ECD = \angle EAM$. Therefore, ω_1, ω_2 are tangent to EM.

Note that the reflection of ω_1 across M must be a circle passing through M and B and tangent to EM. Therefore, the reflection of ω_1 across M is ω_2. □

Claim 4. O_1O_2CD is a parallelogram.

Proof. Notice that $\angle MO_1D + \angle MO_2C = 2\angle MAD + 2\angle MBC = 2(180^\circ - \angle AEB) = 180^\circ$. Thus, $DO_1 \parallel CO_2$. Combined with the fact that ω_1, ω_2 have same radius because they are symmetric across BC, we get that O_1O_2CD is a parallelogram. □

Finally, let P be the reflection of O across AB. Since $\angle O_1OO_2 = 90^\circ$, we have $PM = OM = O_1D = O_2C$. Moreover,

$$\angle PMO_2 = 90^\circ + \angle BMO_2 = 90^\circ + (90^\circ - \angle XMB) = 2\angle CBA = \angle CO_2M.$$

Thus, PMO_2C is an isosceles trapezoid, implying that $PC \parallel O_1O_2$. Combined with the fact that O_1O_2CD is a parallelogram, we have that points P, C, and D are collinear, as desired.

7. **[50]** An infinite sequence a_1, a_2, a_3, $\ldots$ of integers is called *r-close* if every integer appears in the sequence exactly once and $|a_{n+2} - a_{n+1}| \geq r|a_{n+1} - a_n|$ for all integers $n \geq 1$. Determine, with proof, all nonnegative real numbers r for which an r-close sequence exists.

Proposed by: Derek Liu

Answer: All real numbers r such that $0 \leq r < 2$

Solution: First, we provide a construction when $r < 2$. Let k be an integer such that $r < 2 - 2^{-k}$. We can enumerate the integers in the order $\{0, 1, -1, 2, -2, \dots\}$. To construct our sequence, we start with $a_1 = 0$ and go through the enumeration in order. For each number m in the enumeration, if it is already in the sequence, skip it. Otherwise, let the last term in the current sequence be a_n.

In the case where $a_n > m$, pick some positive integer d greater than twice the difference between any two terms in the current sequence. Let $a_{n+1} = a_n + d$, $a_{n+2} = a_{n+1} + 2d$, $a_{n+3} = a_{n+2} + 4d$, and so on, up until $a_{n+k+1} = a_{n+k} + 2^k d$. By definition of d, every previous term in the sequence is within d of a_n, so none of a_{n+1}, a_{n+2}, … could have appeared previously. Note that the successive differences d, $2d$, $4d$, … are each double the last, with d being at least double the previous difference by definition; as $r < 2$, this is valid. Furthermore, we can now let $a_{n+k+2} = m$, as

$$\frac{a_{n+k+1} - m}{a_{n+k+1} - a_{n+k}} = \frac{(2^{k+1} - 1)d + (a_n - m)}{2^k d} > 2 - 2^{-k} > r.$$

Now that we have added m to the sequence, we continue with the enumeration.

In the case where $a_n < m$, pick some integer d in the similar manner but $a_{n+1} = a_n - d$, $a_{n+2} = a_{n+1} - 2d$ and so on. This similar construction allows us to add m to the sequence in this case too.

Since every integer appears in the enumeration, every integer will appear in the sequence. By construction, every integer appears exactly once, as desired.

It remains to show no such sequence exists for $r \geq 2$. Assume for sake of contradiction otherwise, and let $m = a_2 \pm 1$, with sign chosen so that $m \neq a_1$. We know m must appear in the sequence somewhere, say as a_n where $n \geq 3$. Note that $|a_3 - a_2| \geq 2|a_2 - a_1| \geq 2$, and by triangle inequality,

$$\begin{aligned}
|a_n - a_2| &\geq |a_n - a_{n-1}| - |a_{n-1} - a_{n-2}| - \cdots - |a_3 - a_2| \\
&\geq |a_{n-1} - a_{n-2}| - \cdots - |a_3 - a_2| \\
&\geq \dots \\
&\geq |a_3 - a_2| \\
&= 2,
\end{aligned}$$

contradiction.

Thus, such a sequence exists if and only if $r < 2$.

Remark. In the construction, we choose to enumerate the integers in this order: $\{0, 1, -1, 2, -2, \dots\}$. However, any enumerations that include all integers work.

8. **[50]** Let $n \geq 2$ be a positive integer and let $(a_1, \dots, a_n)$ be a permutation of the numbers 1, 2, …, n. Marin makes a single move on this permutation by performing the following steps:

- First, he chooses an integer $1 \leq k \leq \lfloor n/2 \rfloor$. This integer can be different for different moves.
- Then, he picks two nonintersecting sets $I = \{i_1, i_2, \dots, i_k\}$ and $J = \{j_1, j_2, \dots, j_k\}$ such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$.
- Finally, he swaps the numbers a_{i_s} and a_{j_s} in the current permutation for all integers $1 \leq s \leq k$.

Let $f(n)$ denote the smallest positive integer such that it is possible for Marin to reach $(1, 2, \dots, n)$ from any starting permutation in at most $f(n)$ moves. Prove that $\lceil \log_3 n \rceil \leq f(n) \leq \lceil \log_2 n \rceil$.

Proposed by: Marin Hristov Hristov

Answer: N/A

Solution 1: Upper bound. We'll prove that $f(n) \leq \lceil \log_2 n \rceil$ for all positive integers $n \geq 2$ by strong induction on n. The base cases $f(2) = 1$ and $f(3) = 2$ are trivial and can be checked directly. For the induction step, fix $n > 3$, and let $d = \lfloor n/2 \rfloor$. For a permutation $(a_1, \dots, a_n)$ of the numbers $1, \dots, n$, we define

$$\mathcal{I} = \{i \mid a_i > d,\ 1 \le i \le d\} \quad \text{and} \quad \mathcal{J} = \{j \mid a_j \le d,\ d < j \le n\}.$$

Notice that $|\mathcal{I}| = |\mathcal{J}|$ since if we define $\mathcal{L} = \{\ell \mid a_\ell \le d, 1 \le \ell \le d\}$, then from the definition of $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{L}$ it follows that these are pairwise nonintersecting sets. As a result, we get:

$$|\mathcal{I} \cup \mathcal{L}| = |\{i \mid 1 \le i \le d\}| = d = |\{i \mid a_i \le d\}| = |\mathcal{J} \cup \mathcal{L}|.$$

Therefore, $|\mathcal{I}| = d - |\mathcal{L}| = |\mathcal{J}|$, and this will be our choice of k for the first move (if $|\mathcal{I}| = |\mathcal{J}| = 0$, we simply don't make a move at this stage). Since $\mathcal{I} \cap \mathcal{J} = \varnothing$ from above, this value of k is allowed.

For the newly formed permutation $(a'_1, a'_2, \dots, a'_n)$ we have that $(a'_1, a'_2, \dots, a'_d)$ is a permutation of $\{1, 2, \dots, d\}$ and $(a'_{d+1}, \dots, a'_n)$ is a permutation of $\{d+1, \dots, n\}$.

What remains is to notice that after this point, if we want to make a move $(k_1, \mathcal{I}_1, \mathcal{J}_1)$ on a permutation of the numbers $1, \dots, d$, and a move $(k_2, \mathcal{I}_2, \mathcal{J}_2)$ on the numbers $d+1, \dots, n$ (by interpreting these numbers the same as $1, \dots, n-d$), then we would be able to combine them into a single move $(k_1 + k_2, \mathcal{I}_1 \cup \mathcal{I}_2, \mathcal{J}_1 \cup \mathcal{J}_2)$ over the n-element permutation since the structural separation of the two permutations from above. Therefore, we can use the induction hypothesis for $d = \lfloor n/2 \rfloor$ and $n - d = \lceil n/2 \rceil$. In this way, we can always reach $(1, 2, \dots, n)$ using at most

$$1 + \max\{f(d), f(n-d)\} \le 1 + \max\{\lceil \log_2 \lfloor n/2 \rfloor \rceil, \lceil \log_2 \lceil n/2 \rceil \rceil\} = \lceil \log_2 n \rceil$$

moves. Thus, we have proven that $f(n) \le \lceil \log_2 n \rceil$ for all integers $n \ge 2$, as desired. □

Here is also an example of the above procedure for $n = 8$. In every permutation, the elements of $\mathcal{I}$ are bolded, and the elements of the corresponding $\mathcal{J}$ are underlined.

$$(\mathbf{8}, 4, \mathbf{5}, \mathbf{6}, \underline{2}, \underline{1}, \underline{3}, 7) \to (2, \mathbf{4}, \underline{1}, 3, \mathbf{8}, 5, \underline{6}, 7) \to (\mathbf{2}, \underline{1}, \mathbf{4}, \underline{3}, \mathbf{6}, \underline{5}, \mathbf{8}, \underline{7}) \to (1, 2, 3, 4, 5, 6, 7, 8).$$

Lower bound. We will now prove that $f(n) \ge \lceil \log_3 n \rceil$ for all positive integers $n > 1$.

Let us call a set of indices I *troublesome*, if $a_{i_1} > a_{i_2}$ for all indices $i_1 < i_2 \in I$. Consider the starting permutation $(n, n-1, n-2, \dots, 2, 1)$ and note that the set of all indices is troublesome. The key observation we will use is that if a permutation has a troublesome set T, then after any move, the resulting permutation has a troublesome set of size $|T'| \ge |T|/3$.

Indeed, we can notice that $T \cap I$, $T \cap J$ and $T \setminus (I \cup J)$ are nonintersecting troublesome subsets of T. Therefore, at least one of them contains at least $\frac{1}{3}|T|$ elements. Therefore, the size of the largest troublesome subset decreases by a factor of at most 3 after each move. Since $(1, 2, \dots, n)$ does not contain a troublesome set with more than one element, we can conclude that $f(n) \ge \lceil \log_3 n \rceil$ for all integers $n > 1$, as desired.

Solution 2: We first show the upper bound of $\lceil \log_2 n \rceil$. To sort the permutation, consider a recursive routine $f(L, R)$ that sorts $a_L, \dots, a_R$. Let $M = \lfloor (L+R)/2 \rfloor$ be the midpoint of this subarray. Notice that if k elements in $a_L, \dots, a_M$ should have been in $a_{M+1}, \dots, a_R$ after sorting, there are also k elements in $a_{M+1}, \dots, a_R$ that should be in the first half after sorting.

So, we can just let $f(L, R)$ swap the violating elements in the first half with the violating elements in the last half, and recurse down on $f(L, M)$ and $f(M+1, R)$. Notice that all the swaps on the same level can be done in one choice of I, J. This will complete in $\lceil \log_2 n \rceil$ moves.

To show the lower bound of $\lceil \log_3 n \rceil$, denote $L(x)$ for sequence of integers x to be the length of the longest increasing subsequence (LIS) of x.

Claim 1. If a, b, c are three sequences of not necessarily the same length, consider the set of all sequences S formed by interleaving them (keeping order within a, b, c the same.) Then, the max of $L(x)$ over $x \in S$ is at most three times its minimum.

Proof. This follows from noting that

$$\min_{x \in S}(x) \geq \max(L(a), L(b), L(c)) \geq \frac{L(a)+L(b)+L(c)}{3} \geq \frac{\max_{x\in S}(x)}{3}.$$

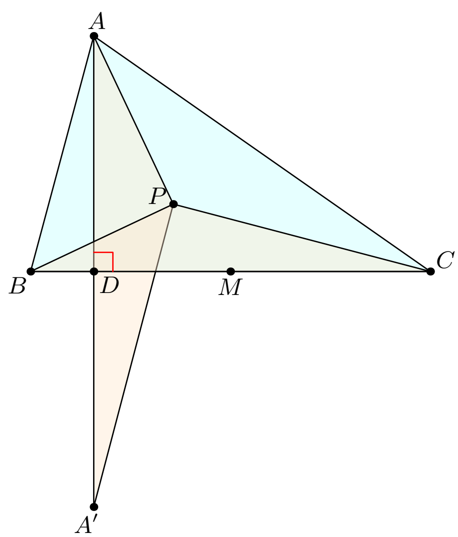
□

9. **[55]** Let ABC be an acute scalene triangle. Let D be the foot of the altitude from A to $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. There exists a unique point P strictly inside triangle ABC such that $\angle DPM = 90^\circ$ and $PB/PC = AB/AC$. Prove that $\angle BPC = 180^\circ - |\angle ABC - \angle ACB|$.

   Proposed by: Aprameya Tripathy

   Answer:

   Solution 1:



Without loss of generality, assume that $AB < AC$. Let A' be the reflection of A across BC. We claim that P be the center of spiral similarity that sends AA' to BC. We show that this P satisfied the required condition. First, note that since $\triangle PAB \sim \triangle PA'C$, we get that

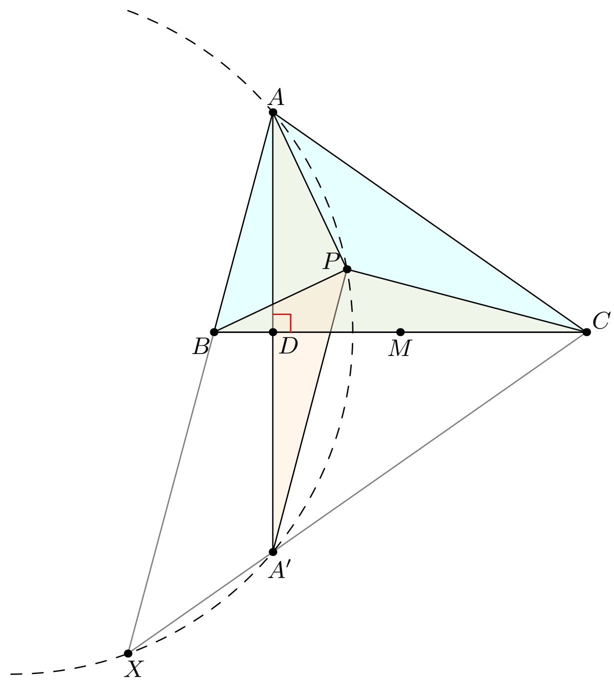
$$\frac{PB}{PC} = \frac{AB}{A'C} = \frac{AB}{AC},$$

verifying the Apollonius circle condition.

Next, we show that $\angle DPM = 90^\circ$. To do this, note that the spiral similarity send midpoint D of AA' to midpoint M of BC. Since $AA' \perp BC$, this spiral similarity is a 90°-rotation, so combining with the previous sentence gives $\angle DPM = 90^\circ$ as desired.

To show the conclusion, we note that since $\triangle PAB \sim \triangle PA'C$, we get that $\angle BPC$ is equal to 180° minus the angle between AB and $A'C$. The angle between AB and $A'C$ is $\angle B - \angle C$, so we have that $\angle BPC = 180^\circ - \angle B + \angle C$, as desired.

Solution 2:



The following is a variant of the first solution.

Let A' be the reflection of A across BC. Recall that the locus of point T such that $\frac{BT}{TC} = \frac{BA}{BC}$ is the Apollonius circle Ω. The key claim is the following.

Claim 1. If lines AB and $A'C$ meet at X, then X lies on Ω.

We present two proofs of the claim.

Proof 1. By angle bisector theorem on $\triangle ACX$, we have that $\frac{BA}{BX} = \frac{CA}{CX}$, which implies that $\frac{BA}{CA} = \frac{BX}{CX}$, so X lies on Ω. $\square$

Proof 2. Let the angle bisector of $\angle BAC$ meets BC at D. By angle bisector theorem, $\frac{BD}{DC} = \frac{BA}{AC}$, so D lies on Ω. We now do angle chasing:

$$\begin{aligned}
\angle ADA' &= 2\angle ADB \\
&= 2\left(180^\circ - \frac{\angle A}{2} - \angle B\right) \\
&= 360^\circ - \angle A - 2\angle B \\
&= 180^\circ - \angle B + \angle C \\
\angle AXA' &= 180^\circ - \angle XBC - \angle XCB \\
&= \angle B - \angle C,
\end{aligned}$$

so $\angle ADA' + \angle AXA' = 180^\circ$, implying that X lies on $\odot(ADA') = \Omega$. $\square$

To finish, redefine point P as the second intersection of $\odot(BXC)$ and Ω. Since $\angle BXC = \angle B - \angle C$, it suffices to show that $\angle DPM = 90^\circ$. To do this, note that since P is the intersection of $\odot(AXA')$ and $\odot(BXC)$, we get that P is the center of spiral similarity that sends AA' to BC. Since $AA' \perp BC$, this spiral similarity must be rotation by 90°. Moreover, this spiral similarity sends D to M. Hence, $\angle DPM = 90^\circ$.

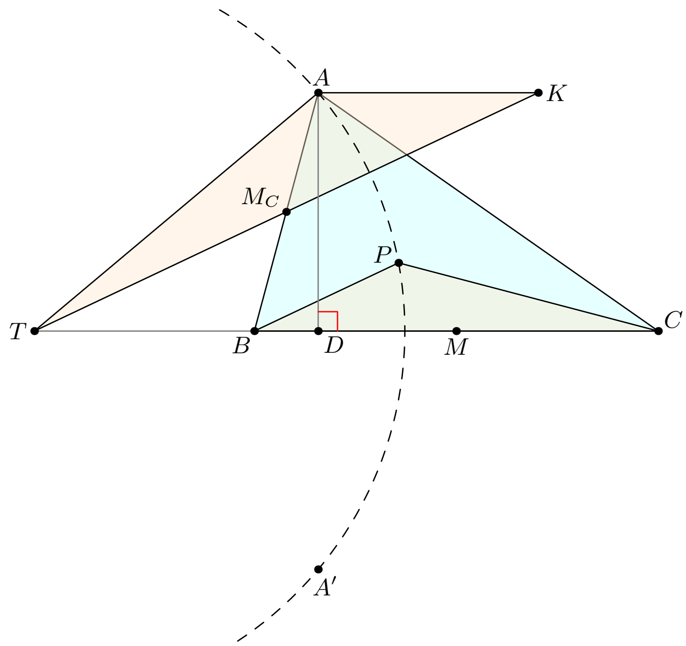
Solution 3: Again, we assume $AB < AC$. The crux of this solution is to claim that $\angle APB = 90^\circ$.

This claim can be obtained by working backward from the conclusion (as shown in after the proof of the claim).

Redefine point P to be the intersection of circle with diameter AB and the Apollonius circle Ω.

Claim 2. With the new definition of P, we have that $\angle DPM = 90^\circ$.

Proof. Let T be the center of Ω, which is well-known to be the intersection of the tangent at A of $\odot(ABC)$ and line BC. Let M_C be the midpoint of AB. Notice that $AP \perp TM_C$, so $TM_C \parallel BP$.



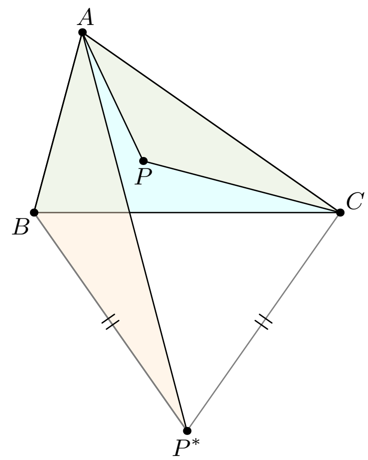
Construct the parallelogram $ATBK$. Then since $TK \parallel BP$ and $AK \parallel BC$, we get that $\angle AKT = \angle PBC$. Moreover, from $\triangle TBA \sim \triangle TAC$, we get that $\frac{PB}{PC} = \frac{AB}{AC} = \frac{TA}{TB} = \frac{TA}{AK}$. The previous two sentences give $\triangle KAT \sim \triangle BPC$. The corresponding midpoint of sides are M_C and M. Hence, $\angle BPM = \angle M_C AK = 180^\circ - \angle B$.

Finally, since $ABPD$ is cyclic, we get that $\angle BPD = \angle BAD = 90^\circ - \angle B$. Combining with the previous paragraph gives $\angle DPM = 90^\circ$. $\square$

We now prove the conclusion. This part can be cited from IMO 1996 P2, but we reprove it from scratch to make the solution self-contained. Perform the inversion at A with radius $\sqrt{AB \cdot AC}$ followed by reflection across angle bisector of $\angle BAC$. Suppose that this inversion sends point P to P^*. First, the similarities $\triangle ABP \sim \triangle AP^*C$ and $\triangle ACP \sim \triangle AP^*B$ gives

$$\frac{CP^*}{AP^*} = \frac{BP}{BA} = \frac{CP}{CA} = \frac{BP^*}{AP^*},$$

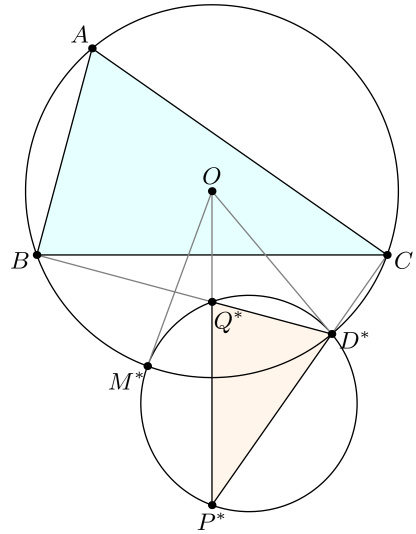
so $BP^* = CP^*$. Thus, P^* lies on the perpendicular bisector of BC.



Thus, $\angle P^*BC = \angle P^*CB$. Moreover, from $\angle APB = 90^\circ$, we have $\angle ACP^* = 90^\circ$. Thus, $\angle P^*CB = 90^\circ - \angle C$, implying that $\angle ACP = \angle ABP^* = 90^\circ + \angle B - \angle C$. Hence, $\angle BPC = 180^\circ - \angle C + \angle B$, as desired.

Solution 4: Again, we assume $AB < AC$. We prove the converse: that if $\frac{BP}{PC} = \frac{BA}{AC}$ and $\angle BPC = 180^\circ - \angle B + \angle C$, then $\angle DPM = 90^\circ$.

Perform the inversion at A with radius $\sqrt{AB \cdot AC}$ followed by reflection across angle bisector of $\angle BAC$, and denote the images of U by U^*. Notice that D^* is an antipode of A with respect to $\odot(ABC)$ and M^* is the point such that $ABCM^*$ is a cyclic harmonic quadrilateral.



We now locate P^*. To do this, we have to translate the two condition. First, the similarities $\triangle ABP \sim \triangle AP^*C$ and $\triangle ACP \sim \triangle AP^*B$ gives

$$\frac{CP^*}{AP^*} = \frac{BP}{BA} = \frac{CP}{CA} = \frac{BP^*}{AP^*},$$

so $BP^* = CP^*$. Thus, P^* lies on the perpendicular bisector of BC.

Now, we translate the condition $\angle BPC = 180^\circ - \angle B + \angle C$. Note that

$$\begin{aligned}\angle BP^*C &= \angle BP^*A + \angle AP^*C \\ &= \angle ABP + \angle ACP \\ &= \angle BPC - \angle A \\ &= 180^\circ - \angle B + \angle C - \angle A = 2\angle C,\end{aligned}$$

which implies that $\angle P^*BC = \angle P^*CB = 90^\circ - \angle C$. In particular, P^*, C, D^* are collinear.

We have to show that AD is tangent to $\odot(DPM)$, or equivalently, AD^* is tangent to $\odot(D^*P^*M^*)$. To do this, we let the perpendicular bisector of BC meet BD^* and point Q^*. Since $\triangle D^*P^*Q^* \sim \triangle ABC$ with corresponding sides perpendicular, we deduce that AD^* is tangent to $\odot(D^*P^*Q^*)$. Therefore, it suffices to show that M^* lies on this circle. To do this, let the $\odot(D^*P^*Q^*)$ meet $\odot(ABC)$ again at K. Since $OD^* = OK$, we get that OK is tangent to $\odot(D^*P^*Q^*)$. Thus, $\odot(D^*P^*Q^*K)$ is a harmonic quadrilateral, so $D^*K \perp AM^*$, so $K = M^*$. Hence, we are done.

10. **[55]** Prove that there exists a real constant M such that for every prime $p \geq M$ and any positive integer $2 \leq m \leq p-1$, there exist positive integers a and b such that $m \leq a \leq 1.01m$, $p^{0.99} \leq b \leq p$, and p divides $ab - 1$.

Proposed by: Pitchayut Saengrungkongka

Answer:

Solution: Let $f(k) = k^{-1} \bmod p$, so $p \mid kf(k) - 1$.

First, we note that since $p \mid mf(m) - 1$, we get that for all $m \geq 2$, $mf(m) \geq p + 1$, so $f(m) \geq \frac{p+1}{m}$. In particular, if $m < p^{0.01}$, then $a = m$ and $b = f(m)$ works. Henceforth, we assume that $m > p^{0.01}$. In particular, $m > 1000$ by taking M to be large enough.

Assume for contradiction that $f(a) \leq p^{0.99}$ for all $a \in [m, 1.01m]$. Consider the quotients $\frac{af(a)-1}{p}$ as a ranges across the interval $[m, 1.01m]$. There are $0.01m - 1 > 0.005m$ numbers, each has size at most

$$\frac{af(a) - 1}{p} \leq \frac{1.01m \cdot p^{0.99}}{p} < \frac{2m}{p^{0.01}}.$$

Therefore, there exists k such that $\frac{af(a)-1}{p} = k$ for at least $0.001p^{0.01}$ values $a \in [m, 1.01m]$. Such a must divide $pk + 1$, so we deduce that $pk + 1$ has more than $0.001p^{0.01}$ divisors. Since $pk + 1 < p^2$, the following lemma gives a contradiction (by taking $\varepsilon < 0.01$).

Lemma 1. For any real number $\varepsilon > 0$, there exists a constant M (possibly depending on ε) such that for all $n > M$, the number of divisor of n is at most n^ε.

Proof. Enumerate all prime numbers as $p_1 < p_2 < \dots$. Let $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, so the number of divisors is $(e_1 + 1)(e_2 + 1) \dots (e_k + 1)$. We observe that

- If $p_i > 10^{1/\varepsilon}$, then $e_i + 1 \leq 10^{e_i} < p_i^{0.5\varepsilon e_i}$.
- Otherwise, there exists a constant C_i such that $e_i + 1 < C_i p_i^{0.5\varepsilon e_i}$ for all $e_i \geq 0$.

Suppose that m is the largest integer such that $p_m \leq 10^{1/\varepsilon}$. Then we have that the number of divisors of n is

$$(e_1 + 1)(e_2 + 1) \dots (e_k + 1) \leq C_1 C_2 \dots C_m p_1^{0.5\varepsilon e_1} p_2^{\varepsilon e_2} \dots p_k^{0.5\varepsilon e_k} = C_1 C_2 \dots C_m n^{0.5\varepsilon},$$

so picking $M > (C_1 C_2 \dots C_m)^{2/\varepsilon}$ concludes. $\square$