HMMT HMIC 2026

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1. **[6]** Let m and n be even positive integers. Suppose we have a tiling of an $m \times n$ rectangle with $mn/2$ non-overlapping dominoes (1×2 or 2×1). Prove there exists another tiling of the rectangle with dominoes that shares no dominoes with the original tiling.

   (Two tilings are said to *share a domino* if there exists two cells of the rectangle which are covered by a single domino in both tilings.)

   Proposed by: Derek Liu

   Solution: Subdivide the $m \times n$ rectangle into 2×2 squares. Within each square, the original tiling cannot contain both a horizontal domino and a vertical domino, as any horizontal domino intersects any vertical domino in a 2×2 square. Thus, we can construct a tiling where each 2×2 square is tiled with two dominoes, neither of which are shared with the original tiling, as desired.

2. **[7]** Prove that there exist infinitely many positive integers n such that $n + 100$ divides $3^n - 2^n - 1$.

   Proposed by: Pitchayut Saengrungkongka

   Solution: Note that when $n + 100 = 101p$ for some prime $p \equiv 1 \pmod{100}$ larger than 101,

   $$3^n - 2^n - 1 \equiv 3^1 - 2^1 - 1 \equiv 0 \pmod{p},$$

   as $n \equiv 101p - 100 \equiv 1 \pmod{p-1}$, and

   $$3^n - 2^n - 1 \equiv 3^1 - 2^1 - 1 \equiv 0 \pmod{101},$$

   as $n \equiv 101p - 100 \equiv 1 \pmod{100}$. Thus, $n + 100$ divides $3^n - 2^n - 1$. Since there are infinitely many primes $p \equiv 1 \pmod{100}$ by Dirichlet's theorem, we are done.

3. **[8]** Let ABC be a scalene triangle with circumcenter O and symmedian point K. Points X and Y lie on the interiors of sides $\overline{AC}$ and $\overline{AB}$, respectively, such that $XY \parallel BC$. Suppose that the circumcircles of triangles AXB and AYC meet segment $\overline{BC}$ at points $D \neq B$ and $E \neq C$, respectively. Prove that D, E, X, and Y lie on a circle whose center lies on line OK.

   (The symmedian point of triangle ABC is the intersection of the reflections of the B-median and C-median across the angle bisectors of $\angle ABC$ and $\angle ACB$, respectively.)

   Proposed by: Pitchayut Saengrungkongka

   Solution 1: First, we show that $DEXY$ is cyclic. Indeed, note that

   $$180^\circ - YED = \angle YEB = \angle CAB = \angle XDC = \angle YXD,$$

   so since $XY \parallel BC$, quadrilateral $DEYX$ must be an isosceles trapezoid.

   Let M and N be the midpoints of segments $\overline{EY}$ and $\overline{DX}$ respectively. Since $\triangle BEY$ is similar to $\triangle BAC$, the line from B to M is just the line from B to the midpoint of $\overline{AC}$ reflected over the angle bisector of $\angle BAC$.

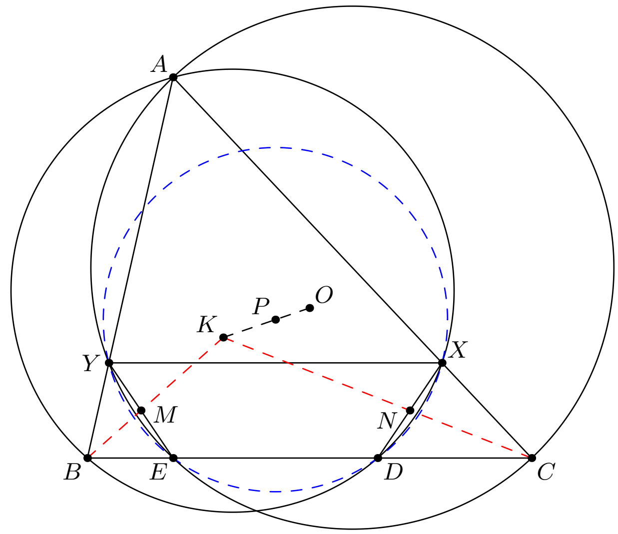
   Thus, points B, M, and K are collinear. Similarly, points C, N, and K are collinear.

   Now, let the center of $(DEXY)$ be P. Then,

   $$\angle MPN = 180^\circ - \measuredangle(DX, EY) = 180^\circ - (180^\circ - \angle BEY - \angle CDY) = 2\angle BAC = \angle BOC.$$

   Thus, since $DEYX$ is an isosceles trapezoid, $PM = PN$, so $\triangle OBC$ and $\triangle PMN$ are similar.

   Finally, note that since $MN \parallel BC$, these two triangles are homothetic. Since $K = BM \cap CN$, K is the center of the homothety, so we are done.



Solution 2: Just as in solution 1, observe that $DEXY$ is cyclic. Let P be its circucmenter. Fix $\triangle AXY$, and move line BC linearly. Note that as we do, line OK also moves linearly.

Now, note that $\angle YXD = \angle XDC = \angle BAC$. As such, XD is tangent to (AXY). Similarly, YE is tangent to (AXY).

Thus, as we move BC linearly, D moves linearly. Then, since $\triangle YPD$ has fixed angles, as D moves linearly, P moves linearly too. Since both OK and P move linearly, we need only show P lies on OK for two choices of line BC.

Firstly, note that when $BC = XY$, $D = X$ and $E = Y$. Then, PX and PY must be perpendicular to the tangents of (AXY) at X and Y, so thus, $P = O$. Then, we trivially have P lies on OK, so we need only consider one more case.

Let the tangents to (AXY) at X and Y meet at T, and consider when BC is the reflection of XY over T. Then, $XYDE$ is a rectangle centered at $T = P$. Thus, T is the midpoint of EY, so since $\triangle BEY \sim \triangle BAC$, B, T, and K are collinear. Similarly, C, T, and K are collinear. Thus, $T = K$, and P lies on OK in this case.

Having checked two cases, we are done.

4. **[10]** Let $n \geq 4$ be an integer. Alice selects n pairwise distinct points $A_1, A_2, \ldots, A_n$ in the plane. In a single query, Bob selects four pairwise distinct integers i, j, k, and ℓ, each between 1 and n, inclusive, and sends them to Alice; Alice then responds "yes" if the lines A_iA_j and A_kA_ℓ are parallel or coincide, and "no" otherwise. In terms of n, determine the minimum number of queries Bob needs to determine whether all n points are collinear, regardless of how the points are positioned.

Proposed by: Derek Liu

Answer: $\boxed{3 \text{ if } n = 4, \text{ and } n-2 \text{ otherwise}}$

Solution If Bob ever gets a "no" response, he knows the points cannot all be collinear, so assume he only gets "yes" responses.

First, we provide a strategy for Bob. If $n = 4$, then Bob asks whether $A_1A_2 \parallel A_3A_4$, $A_1A_3 \parallel A_2A_4$, and $A_1A_4 \parallel A_2A_3$. Assuming they all hold, $A_1A_2A_3A_4$ is a parallelogram, so its diagonals intersect. They are also parallel, so the diagonals coincide, as desired.

If $n > 4$, Bob asks whether $A_1A_2 \parallel A_3A_\ell$ for all $\ell \geq 4$, as well as whether $A_2A_3 \parallel A_4A_5$, for $n-2$ total queries. The first $n-3$ tell Bob A_3 through A_n are collinear, so the last query tells Bob A_2 lies on this line. Then, as $A_1A_2 \parallel A_3A_4$, we know A_1 also lies on this line.

Now, we prove Bob cannot do better. If $n = 4$, then any two queries cannot rule out a parallelogram in some order.

If $n > 4$, we will give Bob the additional info that A_i has x-coordinate i for all i, with which we claim Bob still needs $n-2$ queries. Indeed, every query provides a homogeneous linear equation in the y-coordinates of the points. The space of possible y-coordinates is n-dimensional, and each query reduces this dimension by 1. The y-coordinates for which the points are collinear form a 2-dimensional subspace (parameterized by slope and y-intercept), so at least $n-2$ queries are needed.

Thus our claimed answer is optimal.

5. **[11]** Let n be a positive integer. Determine, in terms of n, the number of ordered n-tuples $(a_1, a_2, \dots, a_n)$ of real numbers such that

- $0 \le a_i < 1$ for all $1 \le i \le n$, and
- $a_1 \cdot 1^k + a_2 \cdot 2^k + \cdots + a_n \cdot n^k$ is an integer for all nonnegative integers k.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\prod_{1 \le i < j \le n}(j-i).}$

Solution 1: We prove by induction on n that the answer is $\prod_{1 \le i < j \le n}(j-i)$. The base case $n = 1$ is trivial. Let $s_k = a_1 \cdot 1^k + a_2 \cdot 2^k + \cdots + a_n \cdot n^k$. We note that

$$s_0,\ s_1,\ s_2,\ \dots \text{ are integers} \iff s_0,\ ns_0 - s_1,\ ns_1 - s_2,\ ns_2 - s_3,\ \dots \text{ are integers.}$$

However, we note that

$$ns_k - s_{k+1} = (n-1)a_1 \cdot 1^k + (n-2)a_2 \cdot 2^k + \cdots + a_{n-1} \cdot (n-1)^k.$$

Therefore, if we let $b_i = (n-i)a_i$, then we want to choose $b_i \in [0, n-i]$ such that $b_1 \cdot 1^k + b_2 \cdot 2^k + \cdots + b_{n-1} \cdot (n-1)^k$ is an integer for all k. This can be done by

- First choose $b_i \in [0, 1]$ such that $b_1 \cdot 1^k + b_2 \cdot 2^k + \cdots + b_{n-1} \cdot (n-1)^k$ is an integer for all k. By induction hypothesis, there are $\prod_{1 \le i < j \le n-1}(j-i)$ ways to do this.
- Next, for each i, we can choose to add one of $0, 1, 2, \dots, n-i-1$, giving $(n-i)$ ways. In total, there are $\prod_{i=1}^{n-1}(n-i)$ ways.
- Finally, for each choice of $b_1, \dots, b_{n-1}$ (hence $a_1, \dots, a_{n-1}$), there is exactly one way to choose a_n so that $s_0 = a_1 + \cdots + a_n$ is an integer.

Thus, there are exactly

$$\prod_{1 \le i < j \le n-1}(j-i) \cdot \prod_{i=1}^{n-1}(n-i) = \prod_{1 \le i < j \le n}(j-i)$$

ways to choose $(a_1, \dots, a_n)$

Solution 2: Let $s_k = a_1 \cdot 1^k + a_2 \cdot 2^k + \cdots + a_n \cdot n^k$.

Claim. If $s_0, s_1, \dots, s_{n-1}$ are integers, then s_k is an integer for all $k \ge 0$.

Proof. Let $c_0, c_1, \dots, c_{n-1}$ be integers such that

$$(x-1)(x-2)\dots(x-n) = x^n - c_{n-1}x^{n-1} - \cdots - c_1x - c_0.$$

In particular, we have that

$$x^k = c_{n-1}x^{k-1} + \cdots + c_1x^{k-n+1} + c_0x^{k-n} \quad \text{for all } x \in \{1, 2, \dots, n\},$$

which implies that

$$s_k = c_{n-1}s_{k-1} + \cdots + c_1s_{k-n+1} + c_0s_{k-n},$$

which implies the claim by an easy induction. □

Let T be the set of tuples $(a_1, \ldots, a_n) \in \mathbb{Q}^n$ such that $s_k = a_1 \cdot 1^k + a_2 \cdot 2^k + \cdots + a_n \cdot n^k$ is an integer for all k. We consider the map of $\mathbb{Z}$-modules

$$\begin{gathered} \phi : T \to \mathbb{Z}^n \\ (a_1, \ldots, a_n) \mapsto (s_0, s_1, \ldots, s_{n-1}), \end{gathered}$$

which is injective because the determinant of Vandemonde matrix is nonzero. Note that $\mathbb{Z}^n$ is a submodule of T, and the number of tuples $a_1, \ldots, a_n$ we are seeking is exactly equal to the index $[T : \mathbb{Z}^n]$. Therefore, by comparing the index, we find that $[T : \mathbb{Z}^n] = [\mathbb{Z}^n : \phi(\mathbb{Z}^n)]$. Thus, the answer is just the density of the lattice $\phi(\mathbb{Z}^n)$ in $\mathbb{Z}^n$. However, this lattice is spanned by vectors

$$\begin{gathered} (1^0, 1^1, 1^2, \ldots, 1^{n-1}) \\ (2^0, 2^1, 2^2, \ldots, 2^{n-1}) \\ \vdots \\ (n^0, n^1, n^2, \ldots, n^{n-1}) \end{gathered}$$

(the i-th one is obtained by plugging in $a_i = 1$ and $a_j = 0$ for all $j \neq i$). Thus, the density of this lattice is

$$\left| \det \begin{pmatrix} 1^0 & 1^1 & \ldots & 1^{n-1} \\ 2^0 & 2^1 & \ldots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n^0 & n^1 & \ldots & n^{n-1} \end{pmatrix} \right| = \prod_{1 \leq i < j \leq n} (j - i),$$

where the determinant is evaluated by noting that it is a Vandermonde matrix. Hence, the answer is $\prod_{1 \leq i < j \leq n} (j - i)$.