HMMT November 2026

November 07, 2026

Guts Round

1. **[5]** Each cell of the following 1×9 grid is labeled with a single-digit nonnegative integer such that any two adjacent cells have a product that is divisible by 6. Compute the least possible sum of all nine numbers in the grid.

?	6	1	?	0	2	?	?	3

Proposed by: Jason Mao

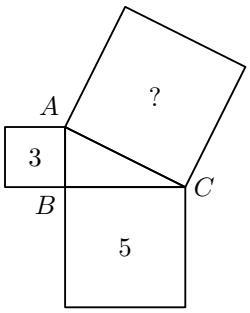
Answer: $\boxed{12}$

Solution: One way to fill the grid is to replace all four unknown cells with a 0.

0	6	1	0	0	2	0	0	3

This is valid because the product of any two adjacent cells is either 0 or 6, both of which are divisible by 6. Furthermore, this must yield the least possible sum because 0 is the least single-digit nonnegative integer. Thus, the least possible sum must be $6 + 1 + 2 + 3 = \boxed{12}$.

2. **[5]** Squares are erected outside of right triangle ABC, as shown below. Given that the square along leg $\overline{AB}$ has perimeter 3, and the square along leg $\overline{BC}$ has perimeter 5, compute the perimeter of the square along hypotenuse $\overline{AC}$.



Proposed by: Jason Mao

Answer: $\boxed{\sqrt{34}}$

Solution: Note that the length of $\overline{AB}$ is $\frac{3}{4}$ and the length of $\overline{BC}$ is $\frac{5}{4}$. By the Pythagorean theorem, the length of $\overline{AC}$ is

$$\sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{5}{4}\right)^2} = \frac{\sqrt{3^2 + 5^2}}{4} = \frac{\sqrt{34}}{4},$$

so the perimeter of the third square is $\boxed{\sqrt{34}}$.

3. **[5]** Compute the number of nine-letter strings consisting of three A's, three B's, and three C's such that each A appears directly next to another A, each B appears directly next to another B, and each C appears directly next to another C.

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Proposed by: Derek Liu

Answer: $\boxed{6}$

Solution: Note that the A's must all form a consecutive block. Indeed, if they were split into two or more blocks, then one of these blocks would have only one A, and that A would not be directly next to another A.

A similar argument applies for the B's and C's, so the only valid strings are composed of the blocks AAA, BBB, and CCC concatenated in any order. There are $3! = \boxed{6}$ such strings, and it is easy to see that all of these strings work.

4. **[6]** Compute the greatest 2-digit positive integer $\underline{ab}$ such that $\underline{ab} + \underline{ba}$ is a perfect square.

Proposed by: Sebastian Attlan

Answer: $\boxed{92}$

Solution: Write $\underline{ab} = 10a + b$ and $\underline{ba} = 10b + a$, so that

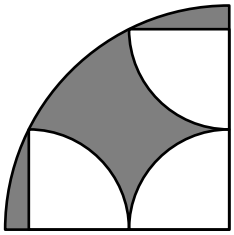
$$\underline{ab} + \underline{ba} = 11a + 11b.$$

Thus, we need $11(a + b)$ be a perfect square. Since a and b range from 0 to 9 with $a \neq 0$, we have $1 \leq a + b \leq 18$, so the only possibility is that $a + b = 11$. To maximize $\underline{ab}$ we pick a to be as large as possible, giving

$$(a, b) = (9, 2) \implies \boxed{\underline{ab} = 92}.$$

This works because $92 + 29 = 121 = 11^2$.

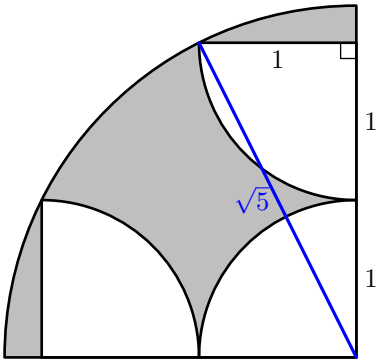
5. **[6]** Three quarter-circles of radius 1 are inscribed in a larger quarter-circle, as shown in the diagram. Compute the area of the shaded region.



Proposed by: Jackson Dryg

Answer: $\boxed{\frac{\pi}{2}}$

Solution: By the Pythagorean Theorem, the radius of the outer quarter-circle is $\sqrt{1^2 + 2^2} = \sqrt{5}$, as shown in the diagram below:



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The area of the outer shaded quarter-circle is therefore

$$\frac{1}{4} \cdot \pi(\sqrt{5})^2 = \frac{5\pi}{4}.$$

Meanwhile, the area of each of the three white quarter-circles is

$$\frac{1}{4} \cdot \pi(1)^2 = \frac{\pi}{4}.$$

Therefore, by subtracting the area of these three white quarter-circles from the ourder shaded quarter-circle, we see that the answer is

$$\frac{5\pi}{4} - 3 \cdot \frac{\pi}{4} = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}.$$

6. **[6]** Snow White and the seven dwarfs are sitting in a circle facing inward. Initially, Snow White holds a basket containing 100 apples, of which 1 is poisoned and the rest are harmless. Every minute, whoever is holding the basket must choose an apple uniformly and independently at random from the basket and eat it, then pass the basket to the person to their left. This process continues until someone eats the poisoned apple.

   Compute the probability that Snow White does **not** eat the poisoned apple.

   Proposed by: Kyle Wu

   Answer: $\boxed{\frac{87}{100}}$

   Solution: Suppose that the process continues even after the poisoned apple is eaten, until all apples are eaten. The probability that Snow White does not eat the poisoned apple does not change.

   Snow White then eats $\lceil \frac{100}{8} \rceil = 13$ apples, and the probability the poisoned apple is among these 13 is $\frac{13}{100}$, since all apples are equally likely to be poisoned. Thus, the probability that Snow White does not eat the poisoned apple is $1 - \frac{13}{100} = \boxed{\frac{87}{100}}$.

7. **[7]** Suppose $P(x)$ is a quadratic polynomial with real coefficients for which $P(10) - P(1) - P(9) = 62$ and $P(10) - P(2) - P(8) = 118$. Compute $P(10) - P(3) - P(7)$.

   Proposed by: Jason Mao

   Answer: $\boxed{158}$

   Solution: We write $P(x)$ in the general form $P(x) = Ax^2 + Bx + C$ for real numbers A, B, and C. If m and n are real numbers, we have

$$\begin{aligned} P(m+n) - P(m) - P(n) &= (A(m+n)^2 + B(m+n) + C) - (Am^2 + Bm + C) - (An^2 + Bn + C) \\ &= A((m+n)^2 - m^2 - n^2) + B(m+n-m-n) + C - C - C \\ &= 2Amn - C. \end{aligned}$$

   Substituting in $m = 1$, $n = 9$ and $m = 2$, $n = 8$ gives

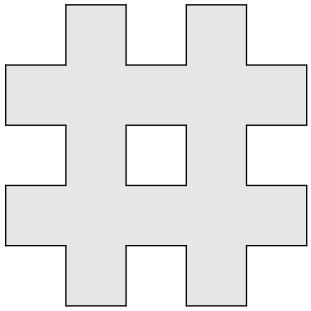
$$\begin{aligned} P(10) - P(1) - P(9) &= 18A - C = 62 \\ P(10) - P(2) - P(8) &= 32A - C = 118, \end{aligned}$$

   which we can solve to get $A = 4$ and $C = 10$. Substituting in $m = 3$ and $n = 7$, we get that

$$P(10) - P(3) - P(7) = 2 \cdot 4 \cdot 3 \cdot 7 - 10 = \boxed{158}.$$

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8. **[7]** Jason splits a 5×5 square into 25 unit squares and removes 9 of them to form the following hashtag shape:

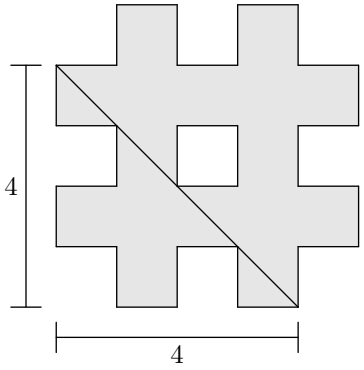


Jason then draws a line segment inside the hashtag (intersections with the boundary are permitted). Compute the maximum possible length of such a segment.

Proposed by: Sam EnMin Huang

Answer: $\boxed{4\sqrt{2}}$

Solution: The answer is $\boxed{4\sqrt{2}}$. The construction is drawn below. The horizontal distance and vertical distance traversed by this segment are both 4, so the length is $\sqrt{4^2+4^2} = 4\sqrt{2}$ by the Pythagorean theorem.



We now show it is impossible to draw a longer segment. Assume for sake of contradiction that a longer segment exists.

Claim 1. This segment must either have one endpoint in the leftmost columnn of squares and one endpoint in the rightmost colum, or one endpoint in the topmost row and one endpoint in the bottommost row.

Proof. As the segment has length greater than $4\sqrt{2}$, either the horizontal distance or the vertical distance traversed by the segment must be greater than 4. (Otherwise, the segment's length would be at most $\sqrt{4^2+4^2} = 4\sqrt{2}$.)

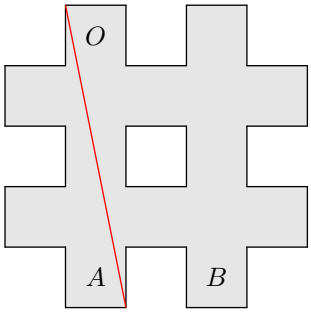
If the horizontal distance is greater than 4, then the segment must pass through squares in 5 different columns. This means the segment must start in the leftmost column of squares and end in the rightmost column of squares. Similarly, if the vertical distance is greater than 4, the segment must start in the topmost row and end in the bottommost row. □

Because the figure is symmetric, we can assume without loss of generality that the segment has one endpoint in the topmost row, and one in the bottommost row. Furthermore, by taking a horizontal reflection, we can assume it has one endpoint in the left square of the topmost row, which is marked

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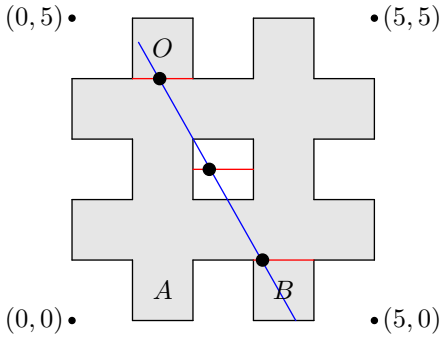
O below. The other endpoint must be in one of the cells of the bottommost row, which are marked A and B below.

If the segment ends in square A, then the longest possible segment passes through opposite corners of O and A, which is the red segment shown below. This segment has length $\sqrt{5^2+1^2}=\sqrt{26}$, which is less than $\sqrt{32}=4\sqrt{2}$.



For the other case where the segment's endpoints are in squares O and B, we have the following claim.

Claim 2. Any segment starting in square O and ending in square B passes through the hole in the center of the hashtag, so it cannot be a valid segment.



Proof. Impose a coordinate system on the hashtag so that the corners of Sam's original square are $(0,0)$, $(0,5)$, $(5,0)$, and $(5,5)$. Observe that any segment contained in the hashtag starting in square O and ending in square B must pass through the lower edge of O and the upper edge of B.

Thus, the segment must pass through a point of the form $(x_1, 4)$ with $1 \le x_1 \le 2$ and a point of the form $(x_2, 1)$ with $3 \le x_2 \le 4$. Taking the average of these two inequalities gives $2 \le \frac{x_1+x_2}{2} \le 3$. The midpoint of these two points (which is also on the segment) is thus $(x_m, 2.5)$ for some $2 \le x_m \le 3$, which is either inside the hole or on one of its vertical edges. Since the line segment cannot be vertical (as $x_1 < x_2$), it necessarily passes through the interior of the hole. □

This exhausts all cases, so the claimed segment of length greater than $4\sqrt{2}$ cannot exist, contradiction. We conclude the maximal length is $\boxed{4\sqrt{2}}$.

9. **[7]** Hannah has 16 colored pencils of different colors and is drawing a tricolor flag with 3 horizontal stripes. The pencils come in 8 pairs of complementary colors. Hannah randomly colors each stripe such that no two stripes have exactly the same color. Compute the probability that no two adjacent stripes have complementary colors.

Proposed by: Rohan Bodke

Answer: $\boxed{\frac{13}{15}}$

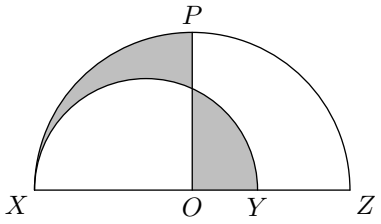
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Solution: Observe that there are always either 0 or 1 complementary pairs since Hannah does not repeat any color.

There are 2 pairs of adjacent stripes. For each such pair, every coloring of the two stripes with distinct colors is equally likely. Given the color of one stripe, exactly 1 out of 15 remaining colors for the second stripe give a complementary pair, so the probability that it is a complementary pair is $\frac{1}{15}$. Thus, the probability that there is a complementary pair is $2 \cdot \frac{1}{15} = \frac{2}{15}$.

It follows that the probability that there is no complementary pair is $1 - \frac{2}{15} = \boxed{\frac{13}{15}}$.

10. **[8]** A semicircle with diameter $\overline{XY}$ is drawn inside of a larger semicircle with diameter $\overline{XZ}$, as shown below. Furthermore, the larger semicircle is divided by radius $\overline{OP}$ into two congruent quarter-circles.



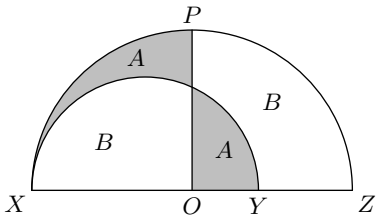
Suppose that the two shaded regions shown above have equal area. Given that $XZ = 10$, compute XY.

Proposed by: Jason Mao

Answer: $\boxed{5\sqrt{2}}$

Solution: Since the areas of the two quarter-circles are the same, the two unshaded regions must also have the same area.

Let the areas of the shaded regions be A and the areas of the unshaded regions be B, as shown below:



Note that the area of the smaller semicircle is $A + B$, and the area of the larger semicircle is $2A + 2B$. Thus, the larger semicircle has twice the area of the smaller semicircle, so the larger semicircle's diameter is $\sqrt{2}$ times as long as the smaller semicircle's diameter. Therefore,

$$10 = XZ = XY \cdot \sqrt{2} \implies XY = \frac{10}{\sqrt{2}} = \boxed{5\sqrt{2}}.$$

11. **[8]** Mark rolls a fair 6-sided die 6 times and writes down his results in order, creating a 6-digit number. Compute the probability that this number is a multiple of 15.

Proposed by: Derek Liu

Answer: $\boxed{\frac{1}{18}}$

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Solution: Note that a number is divisible by 15 if and only if it is divisible by both 3 and 5, so we compute the probability of each of these.

The resulting number is divisible by 5 if and only if the last digit is 5. This happens with probability $\frac{1}{6}$.

Fixing the last five digits, the probability that the resulting number is divisible by 3 is always $\frac{1}{3}$, as exactly 2 of the 6 possibilities for the first die will make the sum of the digits divisible by 3. In particular, this probability is independent of the number being divisible by 5.

Thus, the probability the number is divisible by 15 is

$$\frac{1}{6} \cdot \frac{1}{3} = \boxed{\frac{1}{18}}.$$

12. **[8]** Compute the smallest positive integer $n \geq 5$ such that

- $\binom{n}{0}$ divides $\binom{n}{1}$,
- $\binom{n}{1}$ divides $\binom{n}{2}$,
- $\binom{n}{2}$ divides $\binom{n}{3}$,
- $\binom{n}{3}$ divides $\binom{n}{4}$,
- $\binom{n}{4}$ divides $\binom{n}{5}$.

Proposed by: Darsh Patel

Answer: $\boxed{59}$

Solution: The condition of $\binom{n}{k}$ dividing $\binom{n}{k+1}$ is equivalent to $\frac{\binom{n}{k+1}}{\binom{n}{k}}$ being an integer. Recall that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}.$$

Then we can compute

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{\frac{n(n-1)\dots(n-k+1)(n-k)}{(k+1)!}}{\frac{n(n-1)\dots(n-k+1)}{k!}} = \frac{n-k}{k+1}.$$

Since $\frac{n-k}{k+1} + 1 = \frac{n+1}{k+1}$, this is an integer exactly when $n+1$ is divisible by $k+1$. This holds for $k = 1, 2, 3, 4$, so we need $\operatorname{lcm}(2, 3, 4, 5) = 60$ to divide $n+1$. Then the least possible n is $60 - 1 = \boxed{59}$.

13. **[9]** A circular necklace has 20 red beads and 30 green beads. Stella cuts the necklace into two halves, each with an equal number of beads, and takes one of the halves. The beads are arranged such that across all the ways Stella could have cut the necklace, the minimum possible number of red beads in Stella's half is 6, and the minimum possible number of green beads in Stella's half is N. Compute the maximum possible value of N across all such arrangements of beads.

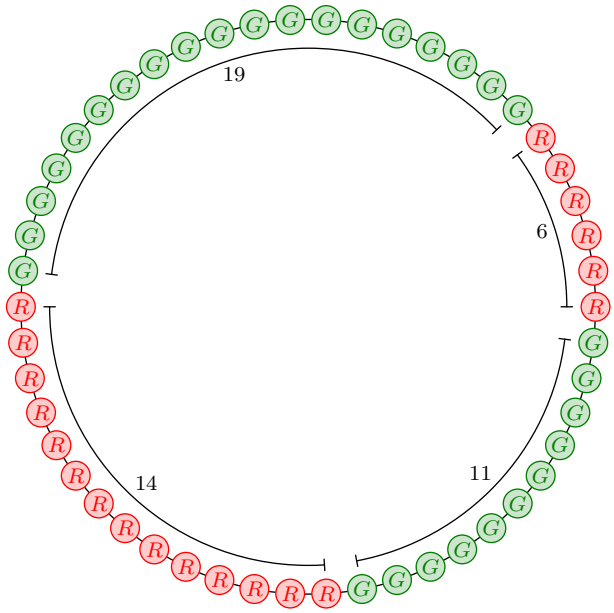
Proposed by: Jason Mao

Answer: $\boxed{11}$

Solution: The maximum number of green beads in a single half is $25 - 6 = 19$. If one half has the maximum number of green beads, the other half must have the minimum number of green beads, so $N = 30 - 19 = \boxed{11}$.

Remark. Such a configuration of beads exists; one example is shown below.

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14. **[9]** Suppose a_1, a_2, a_3, a_4 is a geometric sequence of real numbers satisfying $a_2+a_3 = 20$ and $a_1+a_4 = 26$. Compute $\sqrt[3]{a_1} + \sqrt[3]{a_4}$.

Proposed by: Benny Wang

Answer: $\boxed{\sqrt[3]{86}}$

Solution: Let the common ratio of the geometric sequence be r. Observe that

$$\frac{a_4}{a_2} = \frac{a_4}{a_3} \cdot \frac{a_3}{a_2} = r^2 = \frac{a_2^2}{a_1^2},$$

so $a_1^2 a_4 = a_2^3$. Similarly,

$$\frac{a_3}{a_1} = r^2 = \frac{a_4^2}{a_3^2} \implies a_1 a_4^2 = a_3^3.$$

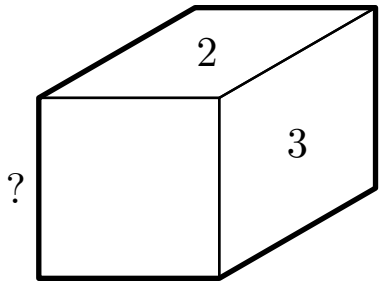
Now,

$$\begin{aligned}(\sqrt[3]{a_1} + \sqrt[3]{a_4})^3 &= a_1 + 3\sqrt[3]{a_1^2 a_4} + 3\sqrt[3]{a_1 a_4^2} + a_4 \\ &= a_1 + 3a_2 + 3a_3 + a_4 \\ &= (a_1 + a_4) + 3 \cdot (a_2 + a_3) \\ &= 26 + 60 = 86,\end{aligned}$$

so $\sqrt[3]{a_1} + \sqrt[3]{a_4} = \boxed{\sqrt[3]{86}}$.

15. **[9]** An equilateral hexagon is divided into a square and two rhombi, as shown below. Given that the rhombi have areas 2 and 3, compute the side length of the hexagon.

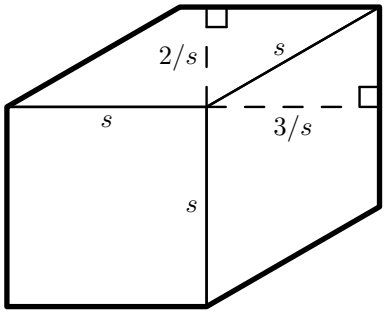
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Proposed by: Jason Mao

Answer: $\boxed{\sqrt[4]{13}}$

Solution: Extend two sides of the square to the boundary of the hexagon, as shown below. The upper rhombus has area 2 and base length s, so its altitude has length $2/s$. Similarly, the right rhombus has altitude length $3/s$.



As shown above, the diagonal of the rectangle with side lengths $2/s$ and $3/s$ has length s. Thus, by the Pythagorean theorem,

$$\left(\frac{2}{s}\right)^2+\left(\frac{3}{s}\right)^2=s^2 \implies 13=s^4 \implies s=\boxed{\sqrt[4]{13}}.$$

16. **[10]** Let N be the smallest positive integer divisible by 2026 with the property that for every positive divisor k of N, the number of positive divisors of k is also a divisor of N. Compute $\frac{N}{2026}$.

Proposed by: Andrew Brahms

Answer: 240

Solution: Note that if a positive integer N has two divisors a and b, then $\text{lcm}(a,b)$ will also be a divisor of N. We repeatedly use this observation to deduce more and more divisors of N as follows.

- Since N is divisible by 2026, which has 4 divisors, 4 is a divisor of N.
- Since 4 has 3 divisors, 3 is a divisor of N. Thus N is divisible by $\text{lcm}(2026,3)=2\cdot3\cdot1013$.
- Since $2\cdot3\cdot1013$ has 8 divisors, 8 is a divisor of N. Thus, N is divisible by $\text{lcm}(2\cdot3\cdot1013,8)=2^3\cdot3\cdot1013$.
- Since $2^3\cdot3\cdot1013$ has $(3+1)\cdot(1+1)\cdot(1+1)=16$ divisors, 16 is a divisor of N.
- Since 16 has 5 divisors, 5 is a divisor of N. Therefore, N is divisible by $\text{lcm}(5,2^3\cdot3\cdot1013)=2^3\cdot3\cdot5\cdot1013$.
- Since $2^3\cdot3\cdot5\cdot1013$ has 32 factors, 32 is a divisor of N.

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Hence, N is divisible by $\text{lcm}(2^4 \cdot 3 \cdot 5 \cdot 1013, 32) = 2^5 \cdot 3 \cdot 5 \cdot 1013$.

We now show that $N = 2^5 \cdot 3 \cdot 5 \cdot 1013$ has the property outlined in the problem statement. Every divisor k of N can be written as $2^a \cdot 3^b \cdot 5^c \cdot 1013^d$, where a is in the set $\{0, 1, 2, 3, 4, 5\}$ and b, c, and d are either 0 or 1, and number of divisors of k will be $(a+1)(b+1)(c+1)(d+1)$. This can be written as $2^r \cdot (a+1)$, where r is equal to the number of $\{b, c, d\}$ that are equal to 1 and is therefore in the set $\{0, 1, 2, 3\}$, while $a+1$ is in the set $\{1, 2, 3, 4, 5, 6\}$.

Therefore, for any divisor k of $N = 2^5 \cdot 3 \cdot 5 \cdot 1013$, the number of divisors of k will be $2^x \cdot 3^y \cdot 5^z$, for nonnegative integers $x \leq 5$, $y \leq 1$, and $z \leq 1$. Hence the number of divisors of k is also a divisor of N, and $N = 2^5 \cdot 3 \cdot 5 \cdot 1013$ satisfies the condition outlined in the problem statement.

Since N must be divisible by $2^5 \cdot 3 \cdot 5 \cdot 1013$ as previously shown, this value of N is the smallest possible. For the answer extraction, $\frac{N}{2026} = 2^4 \cdot 3 \cdot 5 = \boxed{240}$.

17. **[10]** On Monday, Steve writes a single-digit nonzero multiple of 4 on a blackboard. Each following day, Steve inserts a nonzero digit at any position into the number on the blackboard—possibly at the front, in the middle, or at the end—such that the number on the blackboard remains a multiple of 4. Compute the number of 4-digit numbers that could possibly appear on the blackboard on Thursday.

Proposed by: Jacopo Rizzo

Answer: $\boxed{648}$

Solution: We begin with the following claim.

Claim 1. On every day, the rightmost digit is divisible by 4.

Proof. We prove this by induction.

Base case: On the first day, the single digit written on the board is divisible by 4 by assumption.

Induction step: Suppose on a given day that the rightmost digit is x, which is divisible by 4. There are two cases for where Steve can insert a digit on the following day:

- if Steve adds a digit to the front or middle of the number, then the rightmost digit is still x.
- If Steve adds the digit y to the end of the number, then $10x + y$ must be divisible by 4. Since x is divisible by 4, this implies y is divisible by 4, so the rightmost digit remains divisible by 4. □

If the decimal representation of the number on Thursday is $\overline{abcd}$, then $10c + d$ must be divisible by 4. By the claim, d is divisible by 4, so it follows that c is even.

Furthermore, we claim that all numbers $\overline{abcd}$ where d is divisible by 4 and c is even are attainable. Steve could attain them just by writing d on Monday, then c in the front on Tuesday, then b in the front on Wednesday, and finally a in the front on Thursday.

There are then 2 possible choices for d, 4 choices for c, and 9 choices for each of a and b. Thus, the number of 4-digit numbers that could appear on the blackboard on Thursday is $2 \cdot 4 \cdot 9 \cdot 9 = \boxed{648}$.

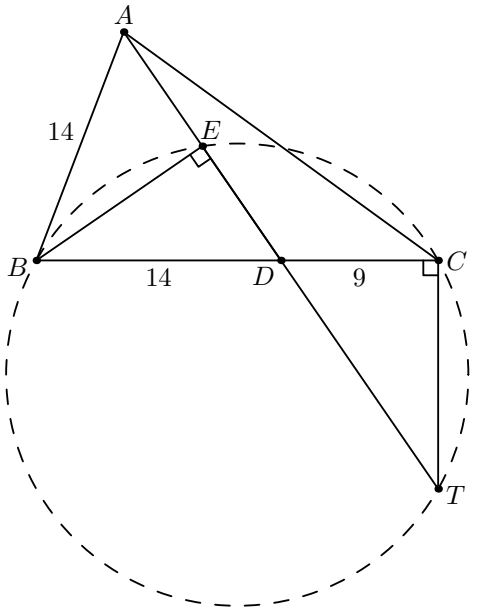
18. **[10]** Point D lies on side $\overline{BC}$ of triangle ABC such that $AB = BD = 14$ and $DC = 9$. Let T be a point on line AD such that $\angle BCT = 90^\circ$ and D lies between A and T. Compute the minimum possible value of AT.

Proposed by: Jason Mao

Answer: $\boxed{12\sqrt{7}}$

Solution:

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Let E be the foot of the perpendicular from B onto $\overline{AD}$. Note that since $\angle BED = \angle TCD = 90°$, quadrilateral $BECT$ is cyclic. By Power of a Point,

$$ED \cdot DT = BD \cdot DC = 14 \cdot 9 = 126.$$

Note that since $\triangle ABD$ is isosceles, $AD = 2ED$. Therefore, by AM-GM, we have

$$\begin{aligned} \frac{AD + DT}{2} &\geq \sqrt{AD \cdot TD} \\ &= \sqrt{2ED \cdot TD} \\ &= \sqrt{252} = 6\sqrt{7}, \end{aligned}$$

so $AT = AD + DT \geq \boxed{12\sqrt{7}}$.

Equality holds if and only if $AD = TD$. We show that this is indeed possible. Select point A so that the foot of the altitude from A to BC is C', the reflection of C over D. This is possible because $BC' = BD - DC = 14 - 9 = 5 < 14 = AB$. In this case, because $\angle AC'D = 90° = \angle TCD$, $\angle ADC' = \angle TDC$, and $C'D = CD$, we have $\triangle ADC' \cong \angle TDC$. This establishes $AD = TD$ as desired.

19. **[11]** Let $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$, $\frac{c_1}{c_2}$, and $\frac{d_1}{d_2}$ be four positive simplified fractions for which

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = \frac{d_1}{d_2}.$$

Given that $a_1 a_2 = 720$, $b_1 b_2 = 108$, and $c_1 c_2 = 250$, compute the minimum possible value of $d_1 d_2$.

Proposed by: Benny Wang

Answer: $\boxed{150}$

Solution 1: Our strategy will be to consider the powers of 2, 3, and 5 in the prime factorizations of the numerators and denominators of all four fractions. We note the prime factorizations of 720, 108, and 250 below.

$$\begin{cases} 720 = 2^4 \cdot 3^2 \cdot 5^1 \\ 108 = 2^2 \cdot 3^3 \cdot 5^0 \\ 250 = 2^1 \cdot 3^0 \cdot 5^3 \end{cases}$$

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Since $a_1a_2 = 720$ and a_1 and a_2 are relatively prime, either the numerator or denominator of the fraction $\frac{a_1}{a_2}$ must contain 2^4 in its prime factorization. Therefore,

- The factor $\frac{a_1}{a_2}$ contributes either a factor of 2^4 or 2^{-4} to the product $\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2}$.
- Analogously, the factor $\frac{b_1}{b_2}$ contributes either a factor of 2^2 or 2^{-2} to the product.
- Finally, the factor $\frac{c_1}{c_2}$ contributes either a factor of 2^1 or 2^{-1} to the product.

So altogether,

- The exponent of 2 in the prime factorization of $\frac{d_1}{d_2} = \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2}$ is $\pm 4 \pm 2 \pm 1$.
  - Thus, the least possible exponent of 2 in the prime factorization of d_1d_2 is $4 - 2 - 1 = 1$.
- Analogously, the exponent of 3 in the prime factorization of $\frac{d_1}{d_2} = \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2}$ is $\pm 2 \pm 3 \pm 0$.
  - Thus, the least possible exponent of 3 in the prime factorization of d_1d_2 is $-2 + 3 + 0 = 1$.
- Finally, the exponent of 5 in the prime factorization of $\frac{d_1}{d_2} = \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2}$ is $\pm 1 \pm 0 \pm 3$.
  - Thus, the least possible exponent of 5 in the prime factorization of d_1d_2 is $-1 + 0 + 3 = 2$.

It follows that $d_1d_2 \geq 2^1 \cdot 3^1 \cdot 5^2 = 150$, which is achievable by the construction shown below:

$$\underbrace{\frac{2^4}{3^2 \cdot 5^1}}_{=\frac{a_1}{a_2}} \cdot \underbrace{\frac{3^3 \cdot 5^0}{2^2}}_{=\frac{b_1}{b_2}} \cdot \underbrace{\frac{5^3}{2^1 \cdot 3^0}}_{=\frac{c_1}{c_2}} = \underbrace{\frac{2^1 \cdot 3^1 \cdot 5^2}{1}}_{=\frac{d_1}{d_2}}.$$

So the minimum possible value of d_1d_2 is $\boxed{150}$.

Solution 2: For any prime number p and rational number x, let $\nu_p(x)$ denote the greatest integer k for which $p^{-k}x$ is an integer. (For example, $\nu_2\left(\frac{1}{12}\right) = -2$, and $\nu_2(720) = \nu_2\left(2^4 \times 3^2 \times 5\right) = 4$.)

Recall the following properties of the function ν_p (which hold for *any* prime p).

(a) The function ν_p satisfies $\nu_p(xy) = \nu_p(x) + \nu_p(y)$ for any $x, y \in \mathbb{Q}$. (Here, $\mathbb{Q}$ is the set of rational numbers.)

(b) For any $x \in \mathbb{Q}$, we have $\nu_p\left(\frac{1}{x}\right) = -\nu_p(x)$.
- *Justification:* Note that $\nu_p(x) + \nu_p\left(\frac{1}{x}\right) = \nu_p(1) = 0$ always.

(c) For any relatively prime $x_1, x_2 \in \mathbb{N}$, we have $\{\nu_p(x_1), \nu_p(x_2)\} = \{\nu_p(x_1x_2), 0\}$ (in some order). Here, $\mathbb{N} = \{1, 2, 3, \dots\}$.
- *Justification:* Since x_1 and x_2 are relatively prime, it cannot be that both $\nu_p(x_1)$ and $\nu_p(x_2)$ are positive, so at least one of them must be 0. Invoking $\nu_p(x_1) + \nu_p(x_2) = \nu_p(x_1x_2)$ implies the conclusion.

Therefore,

$$\begin{aligned}
\pm\nu_p(d_1d_2) &= \nu_p(d_1) - \nu_p(d_2) && \text{by Fact (c)} \\
&= \nu_p\left(d_1/d_2\right) && \text{by Fact (a) and Fact (b)} \\
&= \nu_p\left(a_1/a_2\right) + \nu_p\left(b_1/b_2\right) + \nu_p\left(c_1/c_2\right) && \text{by Fact (a)} \\
&= \left[\nu_p(a_1) - \nu_p(a_2)\right] + \left[\nu_p(b_1) - \nu_p(b_2)\right] + \left[\nu_p(c_1) - \nu_p(c_2)\right] && \text{by Fact (a) and Fact (b)} \\
&= \pm\nu_p(a_1a_2) \pm \nu_p(b_1b_2) \pm \nu_p(c_1c_2) && \text{by Fact (c)}
\end{aligned}$$

To minimize d_1d_2, it is equivalent to minimize the nonnegative value of $\nu_p(d_1d_2)$ for each prime p.

- The above identity for $p = 2$ gives

$$\nu_2(d_1d_2) = |\pm\nu_2(720) \pm \nu_2(108) \pm \nu_2(250)| = |\pm 4 \pm 2 \pm 1| \geq 1.$$

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- The above identity for $p = 3$ gives

$$\nu_3(d_1d_2) = |\pm\nu_3(720) \pm \nu_3(108) \pm \nu_3(250)| = |\pm 2 \pm 3 \pm 0| \geq 1.$$

- The above identity for $p = 5$ gives

$$\nu_5(d_1d_2) = |\pm\nu_5(720) \pm \nu_5(108) \pm \nu_5(250)| = |\pm 1 \pm 0 \pm 3| \geq 2.$$

It then follows that

$$d_1d_2 \geq 2^{\nu_2(d_1d_2)} \cdot 3^{\nu_3(d_1d_2)} \cdot 5^{\nu_5(d_1d_2)} \geq 2^1 \cdot 3^1 \cdot 5^2 = 150.$$

And it is possible to achieve $d_1d_2 = 150$; one possible construction is

$$\underbrace{\frac{2^4}{3^2 \cdot 5^1}}_{=\frac{a_1}{a_2}} \cdot \underbrace{\frac{3^3 \cdot 5^0}{2^2}}_{=\frac{b_1}{b_2}} \cdot \underbrace{\frac{5^3}{2^1 \cdot 3^0}}_{=\frac{c_1}{c_2}} = \underbrace{\frac{2^1 \cdot 3^1 \cdot 5^2}{1}}_{=\frac{d_1}{d_2}}.$$

So the minimum possible value of d_1d_2 is $\boxed{150}$.

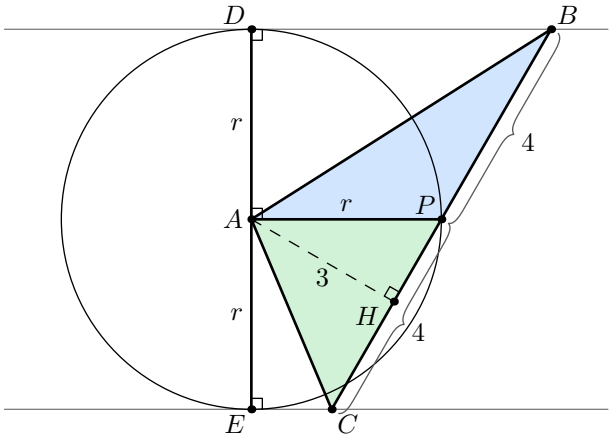
20. **[11]** Let ABC be a triangle with $BC = 8$. Let ω be the circle centered at A passing through the midpoint of side $\overline{BC}$. Suppose that there exist parallel tangent lines to ω through B and C. Given that the distance from A to BC is 3, compute the radius of ω.

Proposed by: James Elias Papaelias

Answer: $\boxed{2\sqrt{3}}$

Solution: Let r be the radius of ω. Let D and E be the tangency points of the tangent lines from B and C respectively to ω. Then, BD is parallel to CE, A is the midpoint of DE, and $DECB$ is a trapezoid with $\angle BDE = \angle DEC = 90°$.

Let H be the foot of the altitude from A to side BC, and let P be the midpoint of BC. Since AP is a midline of $DECB$, we see that AP is also perpendicular to DE, as shown below.



We can split $\triangle ABC$ along AP into two triangles $\triangle APB$ and $\triangle APC$, each with base r and height r. Let $[ABC]$ denote the area of $\triangle ABC$. We see that

$$[ABC] = [ABP] + [APC] = \frac{1}{2}r^2 + \frac{1}{2}r^2 = r^2.$$

At the same time, we have

$$[ABC] = \frac{1}{2}(AH)(BC) = \frac{1}{2}(3)(8) = 12.$$

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Therefore, $r^2 = 12$, so $r = \boxed{2\sqrt{3}}$.

21. **[11]** Compute the number of 10-letter strings of A's and B's that contain the same number of substrings of the form ABB as substrings of the form BBA.

Proposed by: Kira Lewis

Answer: $\boxed{640}$

Solution: Let a string of As and Bs be *good* if it contains the same number of substrings of the form ABB and BBA. The key claim is as follows:

Claim 1. A string of As and Bs is good if and only if one of the following two conditions holds:

(a) The string both starts and ends with BB.

(b) The string neither starts nor ends with BB.

Proof. If there are no As in the string, there are no substrings of the form ABB or BBA. If there are less than two Bs, condition 2 holds, and otherwise condition 1 holds. Therefore the claim holds in this case. Henceforth, we assume that there is at least one A.

Let a *block* be a maximal contiguous substring consisting of only As or only Bs. Let the *associated block* of a substring of the form ABB or BBA be the block of Bs that the BB is contained in. Any associated block has length at least 2 by definition.

Consider a block x of Bs with length at least 2. Then, x must either be bounded between two blocks of As, appear at the beginning of the string, or appear at the end of the string. We consider each case separately.

- If x is bounded between two blocks of As, then x is the associated block of exactly one substring ABB (at the start of x) and one substring BBA (at the end of x).
- If x appears at the beginning of the string, it is followed by a block of As (as we assumed there was at least one A), so x is the associated block of exactly one substring BBA (at the end of x).
- If x appears at the end of the string, it is likewise preceded by a block of As, so x is the associated block of exactly one substring ABB (at the start of x).

Observe that the first case contributes an equal number of substrings of the form ABB and BBA, and the second and third cases can each occur for at most one block. Therefore, the string is good if and only if the second and third case either both occur or both do not occur. Concretely, the string is good if and only if it satisfies one of the following two possibilities:

- It both starts and ends with a block of Bs of length at least 2.
- It neither starts nor ends with a block of Bs of length at least 2.

These possibilities correspond to conditions 1 and 2 in the claim, respectively. This proves the claim. □

There are 2^6 strings of length 10 that satisfy condition 1, as the middle 6 letters can be either A or B, and the other 4 are fixed. For condition 2, there are $2^2 - 1 = 3$ ways to choose the first two letters (to avoid BB), 3 ways to choose the last two letters, and 2^6 ways to choose the middle 6 letters. Therefore, the answer is

$$2^6 \cdot (1 + 3 \cdot 3) = \boxed{640}.$$

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22. **[12]** Compute the sum of the digits in the base 10 representation of

$$\left\lfloor 10^{99} \cdot \frac{100\,008}{700\,007} \right\rfloor.$$

(Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Proposed by: Jason Mao

Answer: $\boxed{440}$

Solution: Since the fraction $\frac{100\,008}{700\,007}$ is less than 1, the problem is equivalent to finding the sum of the first 99 digits after the decimal point in the decimal representation of this fraction.

The key observation is that

$$\frac{100\,008}{700\,007} = \frac{100\,001}{700\,007} + \frac{7}{700\,007} = \frac{1}{7} + \frac{1}{100\,001}.$$

Note that

$$\frac{1}{7} = 0.\overline{142\,857}$$

and

$$\frac{1}{100\,001} = \frac{1}{10^5+1} = \frac{10^5-1}{10^{10}-1} = 0.\overline{00001\,00000} - 0.\overline{00000\,00001}.$$

Hence, adding $\frac{1}{100\,001}$ to $\frac{1}{7}$ is equivalent to preforming the following two operations:

(a) adding 1 to the 5^{th}, 15^{th}, ..., 95^{th} digits after the decimal point, and

(b) subtracting 1 from the 10^{th}, 20^{th}, ..., 90^{th} digits after the decimal point.

Because the digits after the decimal point in $\frac{1}{7}$ are all between 1 and 8 inclusive, neither of these operations will cause any regrouping. Thus, we can compute the answer by finding the sum of the first 99 digits in the decimal representation of $\frac{1}{7}$ and separately considering the effect of adding $\frac{1}{100001}$.

We now compute the answer. Since $99 = 16 \cdot 6 + 3$, the first 99 digits after the decimal point in $\frac{1}{7}$ consist of 16 blocks of 142857, each of which have digit sum 27, followed by the digits 142, which have digit sum 7. Thus the sum of all 99 digits is $16 \cdot 27 + 7 = 439$.

Next, we consider the effect of adding $\frac{1}{100\,001}$. This adds 1 to 10 different digits and subtracts 1 from 9 different digits, so the final answer is $439 + 10 - 9 = \boxed{440}$.

23. **[12]** Call a set S of distinct positive integers *divisorful* if for any ordered triple (a, b, c) consisting of not necessarily distinct elements of S, a divides bc. Compute the least positive integer n such that there exists a divisorful set with exactly 25 elements containing n.

Proposed by: Rohan Bodke

Answer: $\boxed{720}$

Solution: For a positive integer x and prime p, let $\nu_p(x)$ be the largest nonnegative integer k such that p^k divides x. We will use two key properties of ν_p:

- $\nu_p(xy) = \nu_p(x) + \nu_p(y)$ for any prime p and positive integers x and y;
- A positive integer x divides a positive integer y if and only if $\nu_p(x) \leq \nu_p(y)$ for every prime p.

Given a divisorful set S, for each prime p, define $m_p(S)$ to be the minimum value of $\nu_p(x)$ for any x in S, and define $M_p(x)$ to be the maximum value of $\nu_p(x)$ for any x in S. It follows that for any prime p and any positive integer x in S, we have $m_p(S) \leq \nu_p(x) \leq M_p(S)$.

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Claim 1. A set S is divisorful if and only if for every prime p,

$$M_p(S) \leq 2m_p(S). \tag{1}$$

Proof. If $M_p(S) \leq 2m_p(S)$ for all primes p, then for any triple (a, b, c) of elements of S,

$$\nu_p(bc) = \nu_p(b) + \nu_p(c) \geq 2m_p(S) \geq M_p(S) \geq \nu_p(a)$$

for all primes p, so a divides bc. Thus S is divisorful.

On the other hand, if $M_p(S) > 2m_p(S)$ for some prime p, then there must exist a and b in S such that $2\nu_p(b) < \nu_p(a)$. But then a would not divide $b \cdot b$, so S would not be divisorful. □

Claim 2. Let S be a divisorful set with smallest element n. Then:

(a) The set S has at most $\tau(n)$ elements, where $\tau(n)$ is the number of divisors of n.

(b) For any integer $1 \leq N \leq \tau(n)$, there exists a divisorful set with exactly N elements.

Proof. **Part (a):** For each prime p, we have $m_p(S) \leq \nu_p(n)$. And for each element k in S, by Claim 1,

$$m_p(S) \leq \nu_p(k) \leq M_p(S) \leq 2m_p(S).$$

Therefore, there are at most $m_p(S) + 1 \leq \nu_p(n) + 1$ values for $\nu_p(k)$. Considering all primes p, there are at most $\prod_p(\nu_p(n) + 1) = \tau(n)$ possible values for elements k in S, as desired.

Part (b): We construct a divisorful set S' with smallest element n as follows. Choose $m_p(S') = \nu_p(n)$ and $M_p(S') = 2\nu_p(n)$ for every prime p. Then there are exactly $\prod_p(\nu_p(n)+1) = \tau(n)$ possible elements of S'. To start, make S' contain all of them.

Next, if we remove any element of S', the set will remain divisorful. Thus, we can remove elements of S' (that are not n) until we get a divisorful set with smallest element n and size N, as desired. □

By Claim 2, if a divisorful set has 25 elements and smallest element n, then $\tau(n) \geq 25$. Also, if $\tau(n) \geq 25$, then a divisorful set with exactly 25 elements and smallest element n exists. Thus the answer is the smallest n such that $\tau(n) \geq 25$.

Observe that $n^* = 2^4 \cdot 3^2 \cdot 5 = 720$ has 30 divisors and thus works. We claim that no smaller number works.

Note that to minimize n, the values of $\nu_p(n)$ should be non-increasing as p increases. Hence, the smallest n cannot have any prime factor greater than 7, since $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310 > 720$. Similarly, we can show that the minimum n cannot be divisible by 7, since $2 \cdot 3 \cdot 5 \cdot 7$ and $2^2 \cdot 3 \cdot 5 \cdot 7$ have too few divisors, and increasing any other exponent will cause n to exceed 720.

Now, the minimum n cannot be divisible by 5^2, since $2^2 \cdot 3^2 \cdot 5^2 = 900 > 720$, so the smallest n must be of the form 2^u, $2^u \cdot 3^v$, or $2^u \cdot 3^v \cdot 5$. The minimum n with at least 25 divisors in each of these cases is 2^{24}, $2^4 \cdot 3^4 = 1296$, and $2^4 \cdot 3^2 \cdot 5 = 720$, of which 720 is the smallest. Thus, our answer is $\boxed{720}$.

24. **[12]** Gregasaurus has a unit square piece of paper with a red dot in one of its corners. A *fold* consists of the following:

(a) Selecting a line ℓ passing through the interior of the square that divides the square into two regions.

(b) Picking one of the two regions, and reflecting it across ℓ.

Gregasaurus folds the square exactly once. Compute the area of the smallest region R such that the red dot must remain in R, no matter how Gregasaurus folds the square.

Proposed by: Sebastian Attlan

Answer: $\boxed{2\pi + 2}$

Solution: Let the square be $ABCD$ with the red vertex at A. Notice two key claims.

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Claim 1. The point A can be mapped to a point $X \neq A$ with one fold if and only if the perpendicular bisector of $\overline{AX}$ passes through the interior of $ABCD$.

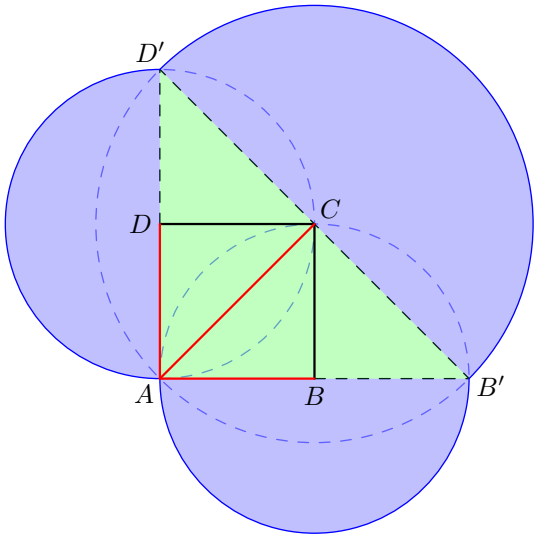
Claim 2. A line ℓ not containing A passes through the interior of $ABCD$ if and only if it passes through the interior of at least one of the segments $\overline{AB}$, $\overline{AC}$, or $\overline{AD}$.

Let S denote the union of the interiors of $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$. Using the first and second claims, as the perpendicular bisector of $\overline{AX}$ for any $X \neq A$ cannot contain point A, we wish to characterize all points X such that the perpendicular bisector of $\overline{AX}$ contains a point on S. In other words, we wish to characterize all points X such that there exists a point P in S such that $AP = PX$. Call such a point *good*.

Claim 3. This set of good points X consists of the union of the following three regions:

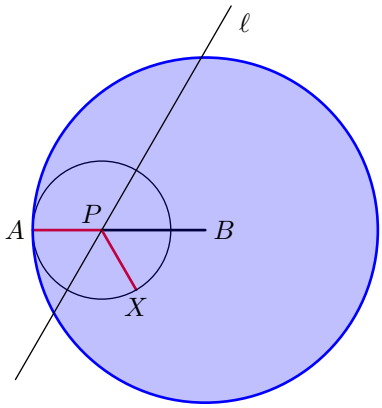
- The (interior of) the circle ω_B centered at B with radius 1.
- The (interior of) the circle ω_C centered at C with radius $\sqrt{2}$.
- The (interior of) the circle ω_D centered at D with radius 1.

The following diagram depicts the set of good points. The red segments form S.



Proof. First we show any good point lies in ω_B, ω_C, or ω_D.

Let X be a good point. Then there exists a point P in S such that $AP = PX$. First, suppose P lies in $\overline{AB}$.



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Then any good point X lies on the circle centered P with radius AP, which lies inside ω_B. Similarly, if P lies in $\overline{AC}$ or $\overline{AD}$, then X lies inside ω_C or ω_D, respectively. We conclude that X lies in ω_B, ω_C, or ω_D, as promised.

Now we show all points in ω_B, ω_C, and ω_D are good. Let X be any point in ω_B; we show that X is good. Consider the circle tangent to ω_B at A passing through X, and let P be the center of this circle. Then P lies on line AB and $AP = PX$, so X is good, as desired. Similar arguments apply for ω_C and ω_D. □

By Claim 3, we wish to find the area of the union of ω_B, ω_C, and ω_D. Let B' and D' be the reflection of A about points B and D, respectively. Then the union consists of the triangle $AB'D'$ along with semicircles of radius 1, $\sqrt{2}$, and 1 centered at B, C, and D, respectively, as shown in the diagram. Triangle $AB'D'$ has area $\frac{1}{2}(2 \cdot 2) = 2$, and the semicircles centered at B, C, and D have areas $\frac{\pi}{2}$, π, and $\frac{\pi}{2}$, respectively. Therefore the answer is $\boxed{2 + 2\pi}$.

25. **[13]** Let $p(x)$ be a polynomial of degree at most 6 such that $p(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ for all $n \in \{2, 3, 4, 5, 6, 7, 8\}$. Compute $p(0)$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{15}{56}}$

Solution: Define the polynomial

$$q(x) = x\Big(p(x) - p(x-1)\Big) - 1.$$

We note the following three observations.

(i) q has roots 3, 4, 5, 6, 7, and 8;

(ii) q has degree at most 6, since $p(x) - p(x-1)$ has degree at most 5;

(iii) $q(0) = -1$.

Observations (i) and (ii) give $q(x) = c(x-3)(x-4)(x-5)\cdots(x-8)$. Thus, (iii) gives

$$\begin{aligned} q(x) &= -\frac{(x-3)(x-4)\cdots(x-8)}{3 \cdot 4 \cdots 8} \\ &= -\frac{2!}{8!}(x-3)(x-4)\cdots(x-8). \end{aligned}$$

Taking $x = 2$ and $x = 1$ in the above equation gives

$$\begin{aligned} 2(p(2) - p(1)) &= 1 - \frac{2!\,6!}{8!} = \frac{27}{28} \\ p(1) - p(0) &= 1 - \frac{2!\,7!}{8!} = \frac{3}{4}, \end{aligned}$$

so we have

$$p(0) = p(2) - \frac{3}{4} - \frac{27}{56} = \left(1 + \frac{1}{2}\right) - \frac{3}{4} - \frac{27}{56} = \boxed{\frac{15}{56}}.$$

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26. **[13]** Sophia has a 5×5 grid of dots, and selects 10 of them uniformly at random. Given that she selects exactly 2 dots in each row and each column, compute the probability that some 4 selected dots form a rectangle with grid-aligned sides.

Proposed by: Kira Lewis

Answer: $\boxed{\frac{5}{17}}$

Solution: Number the columns from 1 to 5. Create a multigraph G with vertices $1, 2, 3, 4, 5$, where for each row, an edge is drawn connecting the column numbers of the two selected dots in the row. (In particular, if multiple rows have the same column numbers selected, multiple edges are drawn between those two column numbers.) Because there are exactly two selected dots in each column, each vertex in G has degree 2. (See the diagrams later in the solution for examples.)

Claim 1. Every connected component of G is a cycle of size greater than 1.

Proof. Since there are no isolated vertices, all connected components have size greater than 1.

Let C be a connected component. We construct a path of G as follows:

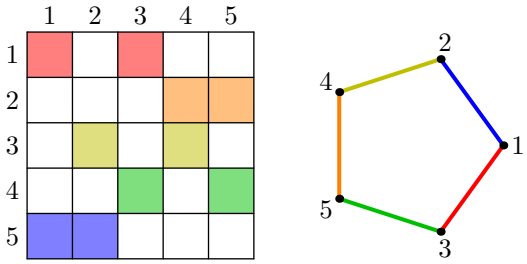
- Start at some vertex v_1 and follow an edge to a vertex v_2.
- At each subsequent vertex, continue along the edge other than the one that was just used.
- Continue until a vertex is repeated, which must occur because C is finite.

We claim the repeated vertex must be v_1. Otherwise, the repeated vertex would have three incident edges, as we entered it twice and left it once. This contradicts the fact that all vertices have degree 2.

Therefore C contains a cycle. Because every vertex has degree 2, the cycle accounts for all the edges of the vertices in it. This finishes the proof of the claim. □

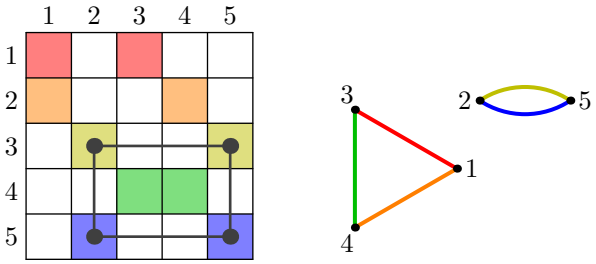
Therefore the connected components of G must consist of either a cycle of length 5, or two cycles of lengths 2 and 3. For some 4 selected dots to form a rectangle, we would need a cycle of length 2, which only occurs in the second case.

In the first case, there are $\binom{4}{2}$ ways to choose the two vertices 1 connects to, and 2 ways to finish the cycle, for a total of 12 possible graphs G. For each graph G, as the edges connect distinct pairs of vertices, there are 5! ways to permute the rows to get a selection of dots.



The first case: G is a 5-cycle

In the second case, there are $\binom{5}{2} = 10$ ways to split the 5 vertices into groups of 2 and 3. From there the cycles are fixed, so there are 10 possible graphs G. For each graph G, as two edges connect the same pair of vertices, there are $\frac{5!}{2}$ ways to permute the rows to get a selection of dots.



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The second case: G is a 3-cycle and a 2-cycle

Therefore the answer is $\frac{10 \cdot 5!/2}{10 \cdot 5!/2 + 12 \cdot 5!} = \frac{5}{5+12} = \boxed{\frac{5}{17}}$.

27. **[13]** Crystal writes two 5-digit positive integers in her journal using every digit exactly once (where leading zeros are not allowed). Compute the maximum possible sum of the digits of the absolute difference between her numbers.

Proposed by: Jason Mao

Answer: $\boxed{40}$

Solution: Say Crystal's numbers are $A > B$. For any positive integer N, let $s(N)$ denote the sum of the digits of N. Furthermore, suppose that when subtracting B from A, a total of R regroupings occur.

Claim 1. We have $s(A - B) = s(A) - s(B) + 9 \cdot R$.

Proof. Consider the typical right-to-left digit-wise subtraction algorithm. At the i^{th} step, we must subtract the i^{th} digit A_i of A from the i^{th} digit B_i of B, possibly applying regrouping.

- If regrouping does not occur, the digit $A_i - B_i$ is written to the final result.
  - This has the effect of adding $A_i - B_i$ to the sum of the digits of $A - B$.
- If regrouping does occur, the digit $A_i - B_i + 10$ is written to the final result, and 1 is subtracted from the digit A_{i+1} to the left of A_i.
  - This has the effect of adding $A_i - B_i + 9$ to the sum of the digits of $A - B$.

So the sum of the digits of $A - B$ is $\left(\sum_{i=1}^{5} A_i - B_i\right) + 9 \cdot R = s(A) - s(B) + 9 \cdot R$, as desired. □

We now optimize $s(A - B)$ for each of the five cases $R \in \{0, 1, 2, 3, 4\}$ for the number of regroupings. Each of these is optimized by taking A to have as many large digits as possible such that the number of regroupings is correct. Since the sum of all ten digits is constant, this minimizes $s(B)$ and thus maximizes $s(A - B)$ by the above claim.

- When $R = 0$, one optimal construction is $98\,765 - 43\,210 = 55\,555$, which has digit sum 25.
- When $R = 1$, one optimal construction is $98\,764 - 32\,105 = 66\,659$, which has digit sum 32.
- When $R = 2$, one optimal construction is $98\,753 - 21\,064 = 77\,689$, which has digit sum 37.
- When $R = 3$, one optimal construction is $98\,642 - 10\,753 = 87\,889$, which has digit sum 40.
- When $R = 4$, one optimal construction is $97\,530 - 18\,642 = 78\,888$, which has digit sum 39.
  - Notice how the pattern breaks at $R = 4$. Attempting to continue the pattern for $R = 4$ yields the construction $97\,531 - 08\,642 = 8\,889$; however, this is not allowed as the smaller of the two numbers has a leading zero and is thus not a five-digit number.

Across all values of R, the greatest digit sum achieved is $\boxed{40}$.

28. **[15]** One hundred players $P_1, P_2, \ldots, P_{100}$ are playing a game. Each player P_i has an unfair coin that lands heads with probability p_i. On the n^{th} turn of the game, player P_n flips their coin, where indices are taken modulo 100. If it lands heads, the game ends immediately and they win. Otherwise, the game continues with the next player's turn.

Given that each player has a $\frac{1}{100}$ probability of winning and that $p_{100} = \frac{6}{7}$, compute p_1.

Proposed by: Jackson Dryg

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Answer: $\boxed{\frac{6}{601}}$

Solution: Let r be the probability that no player has won after 100 turns. Then the probability of P_1 winning is

$$p_1 + rp_1 + r^2p_1 + \cdots = \frac{p_1}{1-r} = \frac{1}{100}. \tag{2}$$

The probability of P_2 winning is

$$(1-p_1)p_2 + r(1-p_1)p_2 + \cdots = \frac{(1-p_1)p_2}{1-r} = \frac{1}{100}. \tag{3}$$

Similarly, we can obtain $\frac{(1-p_1)(1-p_2)p_3}{1-r} = \frac{1}{100}$, and $\frac{(1-p_1)(1-p_2)(1-p_3)p_4}{1-r} = \frac{1}{100}$, and so on.

Dividing equation (2) by equation (3) yields

$$\frac{p_1}{1-p_1} = p_2,$$

and similarly, $\frac{p_i}{1-p_i} = p_{i+1}$ for all $1 \le i \le 99$.

Make the substitution $q_i = \frac{1}{p_i}$. Then

$$q_{i+1} = \frac{1}{p_{i+1}} = \frac{1-p_i}{p_i} = \frac{1-\frac{1}{q_i}}{\frac{1}{q_i}} = q_i - 1.$$

Applying this repeatedly yields

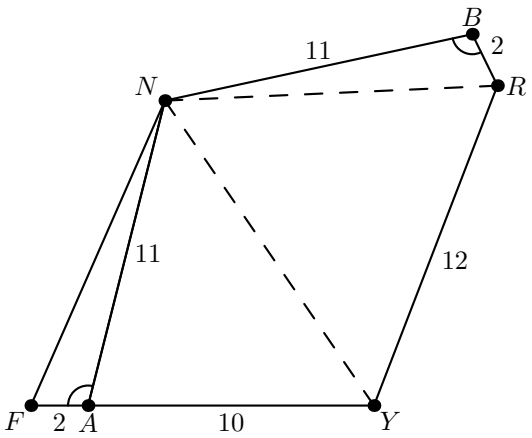
$$q_1 = q_{100} + 99 = \frac{7}{6} + 99 = \frac{601}{6} \quad \Longrightarrow \quad p_1 = \boxed{\frac{6}{601}}.$$

29. **[15]** Convex pentagon $BRYAN$ has area 128 with $\angle A + \angle B = 180^\circ$ and side lengths $BR = 2$, $RY = 12$, $YA = 10$, $AN = 11$ and $NB = 11$. Compute $\sin(\angle A)$.

Proposed by: Benny Wang

Answer: $\boxed{\frac{32}{33}}$

Solution:



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Let F be the point on ray YA past A such that $AF = 2$. By SAS congruence, we have $\triangle BRN \cong \triangle AFN$. Now, by SSS congruence, we have $\triangle RYN \cong \triangle FYN$. Since

$$[YFNR] = [YANR] + [AFN] = [YANR] + [BRN] = [BRYAN] = 128,$$

it follows that

$$[FNY] = \frac{[BRYAN]}{2} = 64,$$

so the distance from N to $\overline{AY}$ is $\frac{32}{3}$. Hence,

$$\sin(\angle A) = \frac{\frac{32}{3}}{11} = \boxed{\frac{32}{33}}.$$

30. **[15]** There exists a polynomial $p(x, y)$ with real coefficients and total degree at most 4 such that $p(m, n)$ is the entry in the m^{th} row and the n^{th} column of the following table.

4	8	*	?
3	5	9	*
2	3	5	9
0	1	2	2

Compute $p(4, 4)$, i.e., the top-right entry in the table.

Proposed by: Derek Liu, Pitchayut Saengrungkongka

Answer: $\boxed{30}$

Solution: Given a polynomial $f(x, y)$, define the finite differences

$$\Delta_x f = f(x + 1, y) - f(x, y) \quad \text{and} \quad \Delta_y f = f(x, y + 1) - f(x, y).$$

Observe that

$$\Delta_x(x^a y^b) = (x + 1)^a y^b - x^a y^b = (ax^{a-1} + \cdots)y^b$$

has degree at most $a + b - 1$ unless $a = b = 0$ (note that if $a = 0$, then $\Delta_x(x^a y^b) = 0$), and likewise with $\Delta_y(x^a y^b)$. Thus, Δ_x and Δ_y both decrease the degree of a polynomial by at least 1, unless the polynomial is constant.

As p has degree at most 4, this means that $q = \Delta_x\Delta_x\Delta_y\Delta_y p$ has degree at most 0, i.e., is constant. Using the given table, we can compute $q(1, 1)$. Then, q must be this value everywhere, so we can reverse the finite differences to compute $p(4, 4)$.

The finite differences are computed below (each table is interpreted as a table of values similarly to the one in the problem statement):

4	8	*	?
3	5	9	*
2	3	5	9
0	1	2	2

$\xrightarrow{\Delta_y}$

4	*	*
2	4	*
1	2	4
1	1	0

$\xrightarrow{\Delta_y}$

*	*
2	*
1	2
0	−1

$\xrightarrow{\Delta_x}$

*	*
1	*
1	3

$\xrightarrow{\Delta_x}$

*	*
0	*

Thus, $q = 0$ everywhere, and we can reverse the finite differences:

4	8	15	30
3	5	9	18
2	3	5	9
0	1	2	2

$\xleftarrow{}$

4	7	15
2	4	9
1	2	4
1	1	0

$\xleftarrow{}$

3	8
2	5
1	2
0	−1

$\xleftarrow{}$

1	3
1	3
1	3

$\xleftarrow{}$

0	0
0	0

Thus, $p(4, 4) = \boxed{30}$.

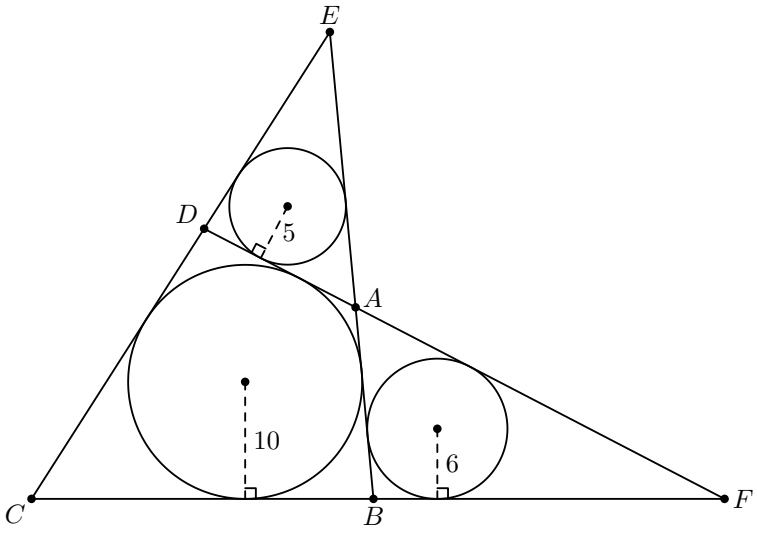
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31. **[17]** Let $ABCD$ be a cyclic quadrilateral, and suppose rays BA and CD meet at E and rays CB and DA meet at F. Suppose the inradii of triangles EDA, EBC, FAB, and FCD are 5, 10, 6, and 10, respectively. Compute the area of $ABCD$.

Proposed by: Benny Wang

Answer: $\boxed{80\sqrt{30}}$

Solution:



Since $ABCD$ is cyclic, we have $\triangle EAD \sim \triangle ECB$, so

$$\frac{DA}{BC} = \frac{5}{10} = \frac{1}{2}.$$

Similarly, $\triangle FAB \sim \triangle FCD$, so

$$\frac{AB}{CD} = \frac{6}{10} = \frac{3}{5}.$$

Note that there is a unique circle of radius 10 tangent to rays CD and CB, so the incircles of $\triangle EBC$ and $\triangle FCD$ coincide. It follows that quadrilateral $ABCD$ has an incircle. By Pitot's theorem,

$$AD + BC = AB + CD.$$

Substituting in our ratios from above, we see

$$\frac{3}{2}BC = AD + BC = AB + CD = \frac{8}{3}AB.$$

Setting $x = \frac{AB}{9}$, it follows that the sides AB, BC, CD, and DA are equal to $9x$, $16x$, $15x$, and $8x$, respectively.

Now, we calculate the area of $ABCD$ in two ways. Firstly, for a quadrilateral with an incircle, we have $A = rs$, so

$$[ABCD] = 10 \cdot 24x = 240x.$$

However, we also know $ABCD$ is a cyclic quadrilateral. One way of calculating the area of a cyclic quadrilateral is Brahmagupta's formula, which gives us the equation

$$\begin{aligned}[ABCD] &= \sqrt{(24x - 9x)(24x - 16x)(24x - 15x)(24x - 8x)} \\ &= 24x^2\sqrt{30}.\end{aligned}$$

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Setting these two quantities equal, it follows that $x = \frac{10}{\sqrt{30}}$, so $[ABCD] = \boxed{80\sqrt{30}}$.

32. **[17]** Mark has a radioactive ball weighing 1 gram. During the n^{th} minute, each radioactive ball has a 1/2 chance of splitting into two radioactive balls, each of weight 2^{-n} grams (regardless of the ball's initial weight or when it was initially formed). After 7 minutes, compute the expected value of the total weight in grams of Mark's radioactive balls.

Proposed by: Derek Liu

Answer: $\boxed{\frac{2123}{8192}}$

Solution 1: Our approach will be to compute the expected number of balls of each weight after n minutes. This will give us their total expected weight by linearity of expectation.

Note that the expected number of balls after n minutes is $\left(\frac{3}{2}\right)^n$, as each ball has a $\frac{1}{2}$ chance to split into two every minute.

We now compute the expected number of balls of weight 2^{-k} after exactly k minutes. Assuming k is positive, each ball after $k-1$ minutes contributes one expected ball of weight 2^{-k}, so the total expected number is $\left(\frac{3}{2}\right)^{k-1}$. (This is true unless $k = 0$, in which case the expected number is just 1.)

After $n \geq k$ minutes, the only way for a ball of weight 2^{-k} to remain is if it formed in the k^{th} minute, then subsequently never split for $n-k$ minutes after that. Each ball of weight 2^{-k} at the k^{th} minute has a $\left(\frac{1}{2}\right)^{n-k}$ probability of not splitting afterwards. Thus, the expected number of balls of weight 2^{-k} after n minutes is $\left(\frac{3}{2}\right)^{k-1}\left(\frac{1}{2}\right)^{n-k}$. (Again, this is true unless $k = 0$, in which case the expected number is just $\left(\frac{1}{2}\right)^n$, the probability the initial ball never split.)

Thus, after n minutes, the expected total weight is

$$\begin{aligned}\left(\frac{1}{2}\right)^n + \sum_{k=1}^{n}\left(\frac{3}{2}\right)^{k-1}\left(\frac{1}{2}\right)^{n-k}2^{-k} &= \left(\frac{1}{2}\right)^n\left(1+\sum_{k=1}^{n}\left(\frac{3}{2}\right)^{k-1}\right)\\ &= \left(\frac{1}{2}\right)^n\left(1+\frac{(3/2)^n-1}{(3/2)-1}\right)\\ &= \frac{3^n-2^{n-1}}{2^{2n-1}}.\end{aligned}$$

When $n = 7$, the expected total weight is

$$\frac{2187-64}{8192} = \boxed{\frac{2123}{8192}}.$$

Solution 2: Let T_n denote the expected total weight after n minutes. We will derive a recursive formula for T_n. Let us consider the two possible outcomes after the first minute.

- If the ball splits into 2 balls of weight $\frac{1}{2}$ in the first minute, then we consider each ball separately. Consider another random process that is the same as the one in the problem, except that all weights are multiplied by $\frac{1}{2}$. The expected total weight after minutes 1, 2, ..., $n-1$ is thus $\frac{1}{2}T_{n-1}$. Now observe that this process is isomorphic to the process where the ball of weight $\frac{1}{2}$ goes through minutes 2, 3, ..., n. Thus, the expected total weight coming from each ball of weight $\frac{1}{2}$ after minute 1 is $\frac{1}{2}T_{n-1}$. Since there are 2 balls of weight $\frac{1}{2}$, by linearity of expectation, the expected value of total weight in this case is T_{n-1}.
- If the ball does not split, then the modified random process in which all weights are multiplied by $\frac{1}{2}$ and lasts for $n-1$ minutes (as in the previous case) gives the same result as starting with a ball of weight 1 and undergoing minutes 2, 3, ..., n, with the **exception** of precisely one case:

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the initial ball never splits, in which case the modified process outputs $\frac{1}{2}$, but the actual resulting weight is 1. This case happens with probability $\frac{1}{2^{n-1}}$, so correcting for that, we get that the expected value of the total weight in this case is

$$\frac{1}{2}T_{n-1} + \frac{1}{2^{n-1}} \cdot \frac{1}{2} = \frac{1}{2}T_{n-1} + \frac{1}{2^n}.$$

Since both cases happens with probability $\frac{1}{2}$, averaging gives

$$T_n = \frac{3}{4}T_{n-1} + \frac{1}{2^{n+1}}.$$

This is an inhomogeneous linear recurrence, so we follow the standard procedure of solving it. The solution to this linear recurrence consists of two parts:

$$T_n = T_n^{(h)} + T_n^{(p)},$$

where $T_n^{(h)}$ satisfies the homogeneous recurrence $T_n^{(h)} = \frac{3}{4}T_{n-1}^{(h)}$ and $T_n^{(p)}$ is a particular solution that satisfies $T_n^{(p)} = \frac{3}{4}T_{n-1}^{(p)} + \frac{1}{2^{n+1}}$, but not necessarily boundary conditions. In particular, any $T_n^{(p)}$ satisfying this recurrence will allow us to solve for T_n, as the sequence $T_n^{(h)} = T_n - T_n^{(p)}$ will then satisfy

$$\left(T_n^{(h)} - \frac{3}{4}T_{n-1}^{(h)}\right) = \left(T_n - \frac{3}{4}T_{n-1}\right) - \left(T_n^{(p)} - \frac{3}{4}T_{n-1}^{(p)}\right) = \frac{1}{2^{n+1}} - \frac{1}{2^{n+1}} = 0.$$

This means $T_n^{(h)}$ satisfies a homogeneous linear recurrence $T_n = \frac{3}{4}T_{n-1}$, which we already know how to solve.

To solve for $T_n^{(p)}$, since we only need one particular solution, we guess that there is a solution of the form $T_n^{(p)} = \frac{B}{2^n}$. This ensures that $T_n^{(p)} - \frac{3}{4}T_{n-1}^{(p)}$ is proportional to $\frac{1}{2^n}$, which is exactly what we want. We use the recurrence to solve for B:

$$\frac{B}{2^n} = \frac{3}{4} \cdot \frac{B}{2^{n-1}} + \frac{1}{2^{n+1}} \implies B = -1.$$

Therefore, $T_n^{(p)} = -\frac{1}{2^n}$ is a particular solution.

From $T_n^{(h)} = \frac{3}{4}T_{n-1}^{(h)}$, we immediately get that $T_n^{(h)} = A\left(\frac{3}{4}\right)^n$ for some constant A. Thus,

$$T_n = T_n^{(h)} + T_n^{(p)} = A\left(\frac{3}{4}\right)^n - \frac{1}{2^n}.$$

Finally, since $T_1 = 1$, we can solve for A:

$$1 = A\left(\frac{3}{4}\right) - \frac{1}{2} \implies A = 2.$$

Thus,

$$T_7 = 2 \cdot \left(\frac{3}{4}\right)^7 - \frac{1}{2^7} = \frac{3^7 - 2^6}{2^{13}} = \boxed{\frac{2123}{8192}}.$$

33. **[17]** The decimal expansion of 1/179 has a repeating period of length 178. Compute the sum of the squares of the digits in this period.

Proposed by: Jackson Dryg

Answer: $\boxed{5049}$

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Solution: Given a real number r and a positive integer n, let $d_n(r)$ denote the n^{th} digit after the decimal point in r. Thus, our goal is to compute

$$\sum_{n=1}^{178} d_n\left(\frac{1}{179}\right)^2.$$

We observe two properties of d_n. For any real number r, positive integer n, and integer m,

$$d_n(r) = d_1(10^{n-1}r) \quad \text{and} \quad d_n(r-m) = d_n(r).$$

Thus,

$$\sum_{n=1}^{178} d_n\left(\frac{1}{179}\right)^2 = \sum_{n=1}^{178} d_1\left(\frac{10^{n-1}}{179}\right)^2.$$

Note that the repeating period of $1/179$ is equal to the order of 10 modulo 179, as $1/179$ repeats every n digits if and only if $179 \mid 10^n - 1$. As given in the problem statement, 10 has order 178 modulo 179. Thus, 1, 10, 10^2, $\ldots$, 10^{177} hit each nonzero residue modulo 179 exactly once. It follows that

$$\sum_{n=1}^{178} d_1\left(\frac{10^{n-1}}{179}\right)^2 = \sum_{k=1}^{178} d_1\left(\frac{k}{179}\right)^2.$$

Now observe that $d_1\left(\frac{k}{179}\right)$ is the unique digit d for which

$$d \le \frac{10k}{179} < d+1.$$

Thus, $d_1\left(\frac{k}{179}\right) = d$ if and only if $k \in \left[\frac{179d}{10}, \frac{179(d+1)}{10}\right)$.

Note that the interval $\left[\frac{179d}{10}, \frac{179(d+1)}{10} + \frac{1}{10}\right)$ always has length exactly 18, so it contains exactly 18 integers. Thus, $\left[\frac{179d}{10}, \frac{179(d+1)}{10}\right)$ contains exactly 18 integers unless there is an integer in $\left[\frac{179(d+1)}{10}, \frac{179(d+1)}{10} + \frac{1}{10}\right)$, in which case it contains 17 integers instead. This is equivalent to saying $\frac{179(d+1)}{10}$ is an integer, which is only true for the digit $d = 9$.

As the sum ranges over $k = 1$ to 178, we also exclude $k = 0$, which would be in the interval for $d = 0$. Thus, $d_1\left(\frac{k}{179}\right) = d$ for 17 values of k if $d = 0$ or 9, and 18 values of k otherwise. The answer is then

$$18(1^2 + \cdots + 9^2) - 9^2 = 18 \cdot \frac{9 \cdot 10 \cdot 19}{6} - 9^2 = 9(570 - 9) = \boxed{5049}.$$

34. **[20]** Krad starts with the number 1. On each step, if his number is n, he replaces it with a random positive divisor of $6n$. Let p be the probability that after 2026 steps, Krad has the number 48. Estimate $1/p$.

Submit a real number E. If the correct answer is A, you will receive $\left\lfloor 20.99e^{-0.02(E-A)^4} \right\rfloor$ points.

Proposed by: Rohan Bodke

Answer: $\boxed{\approx 24e^2 = 177.337346374\ldots}$

Solution: Let $\nu_2(n)$ denote the largest integer k such that 2^k divides n, and define $\nu_3(n)$ similarly. Observe that the values of $\nu_2(n)$ and $\nu_3(n)$ are independent of each other. Furthermore, each of them follow the following random process:

- Start with the number 0.

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- On each step, if the number is m, replace it with a uniformly random number between 0 and $m+1$, inclusive.

Thus, we may estimate the probability that $\nu_2(n) = 4$ and $\nu_3(n) = 1$ after 2026 steps, then multiply the results together.

Let n_i denote Krad's number after i steps. For nonnegative integers i and j, let $p_{i,j}$ denote the probability that $\nu_2(n_i) = j$. (This is also the probability that $\nu_3(n_i) = j$.) For $\nu_2(n_i) = j$ to be true, it must have been the case that $\nu_2(n_{i-1}) \geq j-1$. If this is the case, then there is a $\frac{1}{\nu_2(n_{i-1})+2}$ probability that $\nu_2(n_i) = j$, as there is an equal probability of it being any integer from 0 to $\nu_2(n_{i-1})+1$, inclusive. Therefore, we have the recurrence relation

$$p_{i,j} = \sum_{k=j-1}^{\infty} \frac{p_{i-1,k}}{k+2},$$

with initial conditions $p_{0,0} = 1$ and $p_{0,j} = 0$ for $j \neq 0$. (We will say that $p_{i,-1} = 0$.)

After 2026 steps, we expect the probability of $\nu_2(n) = j$ for any j to be relatively stable, i.e., $p_{2026,j} \approx p_{2027,j}$. We expect $p_{2026,1}$ and $p_{2026,4}$ to be much larger than the discrepancies $p_{2026,j} - p_{2027,j}$, so the values of $1/p_{2026,1}$ and $1/p_{2026,4}$ should also be relatively stable. (Formally, the convergence and decay of discrepancies is guaranteed by the fundamental theorem of Markov chains.) As such, we first compute the long-term steady state of this random process.

Let the *steady-state* sequence $q_0, q_1, \ldots$ be the unique sequence satisfying $q_0 + q_1 + \cdots = 1$ and

$$q_j = \sum_{k=j-1}^{\infty} \frac{q_k}{k+2} \tag{4}$$

for all $j \geq 0$, again setting $q_{-1} = 0$. (We will establish uniqueness below.) Our goal is to compute $\frac{1}{q_1 q_4}$.

We compute the sequence q_j. Plugging $j+1$ and j into (4) and subtracting yields

$$q_j - q_{j+1} = \frac{q_{j-1}}{j+1}. \tag{5}$$

Claim 1. For all $m \geq 0$, we have $q_m = \frac{q_0}{j!}$.

Proof. We proceed by strong induction on m.

Base case: $m \leq 1$. When $m = 0$, the claim is immediate. When $m = 1$, plugging $j = 0$ into (5) yields

$$q_0 - q_1 = \frac{q_{-1}}{1} = 0,$$

so $q_1 = q_0 = \frac{q_0}{1!}$.

Induction step: Suppose that $m \geq 2$ and $q_k = \frac{q_0}{k!}$ for all $k < m$. Plugging $j = m-1$ into (5) yields $q_{m-1} - q_m = \frac{q_{m-2}}{m}$. Using the induction hypothesis, we get

$$\begin{aligned} q_m &= q_{m-1} - \frac{q_{m-2}}{m} \\ &= q_0 \left(\frac{1}{(m-1)!} - \frac{1}{(m-2)! \cdot m} \right) \\ &= \frac{q_0}{(m-2)!} \left(\frac{1}{m-1} - \frac{1}{m} \right) \\ &= \frac{q_0}{m!}, \end{aligned}$$

which finishes the inductive step. □

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Using the above claim and the fact that $q_0 + q_1 + \cdots = 1$, we get

$$1 = q_0 \sum_{j=0}^{\infty} \frac{1}{j!} = q_0 \cdot e.$$

Therefore $q_0 = \frac{1}{e}$, and by the claim, $q_j = \frac{1}{e \cdot j!}$. In particular, the sequence q_j is unique (and so the fundamental theorem of Markov chains applies).

Our final estimate is $\frac{1}{q_1 q_4} = e(24e) = 24e^2$. A crude approximation of $e^2 \approx 2.72^2 \approx 7.4$ yields an estimate of $24 \cdot 7.4 = 177.6$, which is more than close enough for all 20 points.

Remark. The following Python code computes the answer; it only deviates from $24e^2$ due to floating-point issues, which far outweigh the discrepancy between the probabilities after 2026 steps and in steady-state.

```
probs = [1] + [0 for i in range(2026)]
for j in range(20):
    probs[2026] = probs[2025]/2027
    for k in range(2025, 0, -1):
        probs[k] = probs[k+1] + probs[k-1]/(k+1)
    probs[0] = probs[1]

print(1/(probs[4]*probs[1]))
```

35. **[20]** Sam has a standard 6×6 checkerboard with black and white squares. Sam then colors in each white square of the checkerboard with black marker independently with probability $1/2$. Estimate the expected number of contiguous black regions now on Sam's checkerboard.

Submit a real number E. If the correct answer is A, you will receive $\left\lfloor 20.99e^{-12(E-A)^2} \right\rfloor$ points.

(A contiguous black region is a maximal set of black squares satisfying the property that, between any two black squares of the set, there exists a path traversing only through other black squares of the set, where consecutive squares on a path share an edge.)

Proposed by: Charles Zhang

Answer: $\boxed{\frac{932081}{262144} = 3.555606842041015625}$

Solution: We can create a planar graph on the checkerboard, after Sam colors it in, by placing a vertex at the center of each black square and connecting adjacent black squares with an edge. The Euler characteristic formula gives

$$C = V - E + F,$$

where C is the number of connected components (continuous black regions), V is the number of vertices, E is the number of edges, and F is the number of faces (regions enclosed by edges). By considering each square and adjacent pair of squares separately, and using linearity of expectation, we can compute:

$$\mathbb{E}[V] = 18 + \frac{1}{2} \cdot 18 = 27,$$
$$\mathbb{E}[E] = \frac{1}{2}(30 + 30) = 30.$$

Note that the probability of a face with many vertices existing decreases exponentially with the size of the face. Hence, a good lower bound on the expected number of faces is the expected number of 2×2 black squares (cycles with 4 vertices), which is the smallest possible face.

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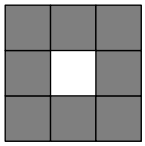
Since there are 25 possible locations for this shape, and each one requires two white squares to be colored in, we have

$$\mathbb{E}[F] \geq \frac{1}{2^2} \cdot 25 = \frac{25}{4}.$$

Hence,

$$\begin{aligned}\mathbb{E}[C] &= \mathbb{E}[V] - \mathbb{E}[E] + \mathbb{E}[F] \\ &\geq 27 - 30 + \frac{25}{4} \\ &= 3.25,\end{aligned}$$

which gives 6 points. To get a better estimate for F, we can also add the number of expected occurrences of the following shape:



This shape can appear in 9 locations, and requires 4 colored white squares and 1 uncolored one, so it appears $\frac{9}{2^5}$ times in expectation. Hence,

$$\begin{aligned}\mathbb{E}[F] &\geq \frac{25}{4} + \frac{9}{32} = 6.53125 \\ \implies \mathbb{E}[C] &\geq 3.53125\end{aligned}$$

which gives 20 points.

36. **[20]** Compute the number of positive integers $N < 10^9$ which satisfy the following two properties:

- N is not a multiple of 2 or 5.
- There does not exist a prime M which has the same number of digits as N and differs from N in at most one decimal place.

Submit a positive integer E. If the correct answer is A, you will receive round $\left(20 \min\left(\frac{A}{E}, \frac{E}{A}\right)^2\right)$ points.

Proposed by: Andrew Lin

Answer: 11155

Solution: Let k be the number of digits of N. On the one hand, there are approximately $9k$ numbers we can obtain by changing one digit of N; on the other hand, the probability of a number being prime scales roughly with $\frac{1}{\log N} \sim \frac{1}{k}$. Consequently, the probability of a 9-digit number satisfying the second property is similar to the probability of a smaller number satisfying it. Thus, we will estimate the probability that a 9-digit number satisfying the first property also satisfies the second, then multiply by $4 \cdot 10^8$ (the number of positive integers up to 10^9 satisfying the first property).

Assume $k = 9$. We first compute the number of ways to change at most one digit of N to produce a number which is not a multiple of 2 or 5. We consider cases on which digit is changed.

- There is 1 way to change no digits (and get back N).

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- There are 3 ways to change the last digit, as it can be any of 1, 3, 7, or 9, except for the last digit of N.
- There are 8 ways to change the first digit, as it can be any digit except 0 and the first digit of N.
- There are 9 ways to change any other digit, as it can be any digit except the corresponding digit of N.

This gives us a total of $3 + 8 + 9 \cdot 7 + 1 = 75$ numbers which differ from N in at most one digit.

A number around N has a probability of around $\frac{1}{\ln N}$ of being prime. Knowing beforehand that the number is not a multiple of 2 or 5, we multiply this estimate by $\frac{2}{1} \cdot \frac{5}{4}$. Assuming the 75 numbers all have independent probabilities of being prime, we obtain an estimate of

$$\left(1 - \frac{5}{2 \ln N}\right)^{75}$$

for the probability N satisfies the second property. This function is increasing and concave down, so to estimate the average value in the interval $[10^8, 10^9]$ we use the value $N = 5 \cdot 10^8$. Then

$$\ln N = 9 \ln 10 - \ln 2 \approx 9(2.3) - 0.7 = 20.$$

Thus, our estimate for the probability is

$$\left(1 - \frac{5}{40}\right)^{75} = \left(\frac{7}{8}\right)^{75} \approx e^{-75/7.5} = e^{-10}.$$

Note that

$$\log_{10}(e^{10}) = \frac{10}{\ln(10)} \approx \frac{10}{2.3} \approx 4.35 \approx 4 + \log(2.5),$$

so our final estimate is around

$$e^{-10}(4 \cdot 10^8) \approx \frac{1}{2.5 \cdot 10^4}(4 \cdot 10^8) = 16000,$$

enough for 10 points.

To get a better estimate, note that if we vary one digit d of N and keep the others constant, we will get 4 copies of one residue modulo 3 (occurring when $d = 0$, 3, 6, and 9) and 3 copies of the other residues modulo 3. As a result, of the 75 numbers which differ from N by in at most one digit, almost exactly one-third of them are divisible by 3. Therefore, we will assume that exactly one-third of our candidates are multiples of 3. (Our prior estimate assumed the probabilities of all 75 numbers being divisible by 3 were independent, which greatly affects the resulting estimate!)

Thus, we assume that of the 75 numbers, only 50 of them are not divisible by 3, and the probability of each of them being prime is $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{1}{\ln N} = \frac{15}{4 \ln N}$. Thus, our new estimate for the probability N satisfies the second property is

$$\left(1 - \frac{15}{4 \cdot 20}\right)^{50} = \left(\frac{13}{16}\right)^{50}.$$

We will also refine our methods to approximate this quantity numerically. We first estimate $\ln \frac{13}{16} = \ln\left(1 - \frac{3}{16}\right)$. We can do this via the Taylor expansion

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right),$$

so

$$\begin{aligned}
\ln\left(\frac{13}{16}\right) &= -\left(\frac{3}{16} + \frac{1}{2}\left(\frac{3}{16}\right)^2 + \frac{1}{3}\left(\frac{3}{16}\right)^3 + \cdots\right) \\
&\approx -\left(0.1875 + \frac{1}{2}(0.036) + \frac{1}{3}(0.007)\right) \\
&\approx -0.208,
\end{aligned}$$

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with error around $\frac{1}{4}\left(\frac{3}{16}\right)^4 \approx 0.0003$.

Thus, $\ln\left(\frac{13}{16}\right)^{50} \approx 50(-0.208) = -10.4$. Note $e^{-11.5} = (e^{2.3})^{-5} \approx 10^{-5}$ and $e^{1.1} \approx 3$, so $e^{-10.4} = e^{1.1-11.5} \approx 3 \cdot 10^{-5}$. Our final estimate is thus

$$(3 \cdot 10^{-5})(4 \cdot 10^8) = 12000,$$

close enough for 17 points.

The prime $p = 3$ is a useful prime to use because the residues of the 10 digits modulo p is roughly equally distributed, meaning the residues of the 75 numbers are roughly equally distributed as well (instead of truly independent). The only other prime for which this is true is $p = 11$, as in this case, 10 of the 11 distinct residues are hit exactly once. (On the other hand, with $p = 7$, roughly half of the residues are hit once, and the rest are hit twice.) Thus, we apply the same correction to our estimate using $p = 11$ as we did using $p = 3$. There are $50 \cdot \frac{10}{11} = \frac{500}{11}$ numbers divisible by neither 3 nor 11, each of which has a $\frac{15}{4 \ln N} \cdot \frac{11}{10} = \frac{33}{8 \ln N}$ probability of being prime, so our estimate for the probability is

$$\left(1 - \frac{33}{8 \cdot 20}\right)^{500/11} = \left(1 - \frac{33}{160}\right)^{500/11}.$$

We estimate the natural log of this quantity:

$$\begin{aligned}
\left(\frac{500}{11}\right) \ln\left(1 - \frac{33}{160}\right) &\approx -\left(\frac{500}{11}\right)\left(\frac{33}{160} + \frac{1}{2}\left(\frac{33}{160}\right)^2 + \frac{1}{3}\left(\frac{33}{160}\right)^3 + \frac{1}{4}\left(\frac{33}{160}\right)^4\right) \\
&\approx -\left(\frac{75}{8} + \frac{2475}{2560} + \frac{75}{8} \cdot \frac{27225}{204800} + 45 \cdot \frac{1}{4}(0.2)^4\right) \\
&\approx -\left(9.375 + \left(1 - \frac{1}{30}\right) + \frac{4}{30} + 0.018\right) \\
&\approx 10.49.
\end{aligned}$$

(The exact estimation method isn't too important; 10.48 and 10.49 will both suffice as estimates.)

Now note that

$$\ln\left(\frac{100}{36}\right) \approx 2(2.3 - 0.693 - 1.1) \approx 1.01 = 10.49 - 5(2.3),$$

so $e^{10.49} \approx \frac{100}{36} \cdot 10^{-5} = \frac{25}{9} \cdot 10^{-5}$. Thus, our final approximation is

$$\left(\frac{25}{9} \cdot 10^{-5}\right)\left(4 \cdot 10^8\right) = \frac{10}{9} \cdot 10^4 \approx 11111,$$

which is close enough for all 20 points.

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