HMMT November 2026

November 07, 2026

Team Round

1. **[20]** Jerry and Sophie are each given a number. They are both told the following information:

   - Their numbers are distinct integers between 2 and 50, inclusive.
   - Jerry's number is an integer power of Sophie's number.

   However, neither of them can determine the other's number given this information. Compute the sum of their numbers.

   Proposed by: Derek Liu

   Answer: $\boxed{18}$

   Solution: Let Jerry's number be j and Sophie's number be s. Let n be the positive integer such that $j = s^n$. Since j and s are distinct, $n > 1$.

   Let t and m be positive integers such that $j = t^m$ and t is as small as possible. Then, $n \mid m$. Thus, m must have at least 2 distinct divisors which are greater than 1 (otherwise Jerry can determine n and thus s).

   Note that $m \leq 5$, as $2^6 > 50$ is too large. The only possible m with enough divisors is $m = 4$. Since $3^4 > 50$, we know $t = 2$.

   Thus, $j = 2^4 = 16$, which is a power of both 2 and 4. If Sophie had 4, she would know Jerry has $4^2 = 16$ because $4^3 > 50$. Thus, $s = 2$, so the answer is $16 + 2 = \boxed{18}$.

2. **[25]** Across all positive integers m and n, compute the least possible integer value of

$$(\sqrt{21} + \sqrt{55})^2 + (\sqrt{m} - \sqrt{n})^2.$$

   Proposed by: Kyle Wu

   Answer: $\boxed{144}$

   Solution: Expanding the given expression yields

$$21 + 2\sqrt{21 \cdot 55} + 55 + m - 2\sqrt{mn} + n.$$

   Because m and n are positive integers, this expression is an integer if and only if $2\sqrt{21 \cdot 55} - 2\sqrt{mn}$ is an integer.

   Claim 1. This condition is satisfied if and only if $mn = 21 \cdot 55$.

   Proof. Clearly, if $mn = 21\sqrt{55}$, then $2\sqrt{21 \cdot 55} - 2\sqrt{mn} = 0$ is an integer.

   Assume for contradiction that $mn \neq 21 \cdot 55$. Then, there exists some nonzero integer k so that $2\sqrt{21 \cdot 55} = 2\sqrt{mn} + k$. Squaring both sides yields

$$4 \cdot 21 \cdot 55 = 4mn + 4k\sqrt{mn} + k^2.$$

   Because $k \neq 0$, we may rearrange this equation to yield

$$2\sqrt{mn} = \frac{4 \cdot 21 \cdot 55 - 4mn - k^2}{2k}.$$

   Therefore, $2\sqrt{mn}$ is rational. Then as $2\sqrt{21 \cdot 55} - 2\sqrt{mn}$ is an integer, $2\sqrt{21 \cdot 55}$ must be rational. But it is clear that $2\sqrt{21 \cdot 55}$ is irrational, so we have a contradiction. □

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The claim implies that m and n are factors of $21 \cdot 55$, as they are positive integers. When $mn = 21 \cdot 55$, the expression is equal to $21 + 55 + m + n$, so it suffices to minimize $m + n$ over all positive integers m and n satisfying $mn = 21 \cdot 55$.

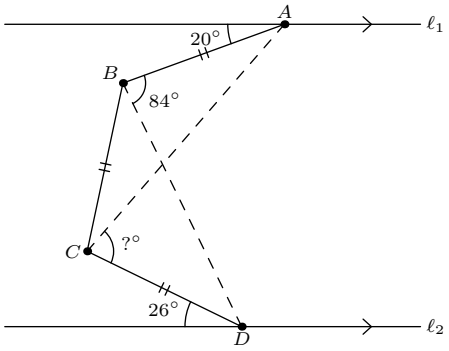
We claim that the minimum value of $m + n$ is 68, which is achieved when $m = 33$ and $n = 35$. To see why $m + n$ cannot be smaller, observe that

$$(m+n)^2 \geq (m+n)^2 - (m-n)^2 = 4mn = 4 \cdot 21 \cdot 55 > 67^2.$$

Since $m + n$ is a positive integer, we must have $m + n \geq 68$.

Therefore, the answer is $21 + 55 + 68 = \boxed{144}$.

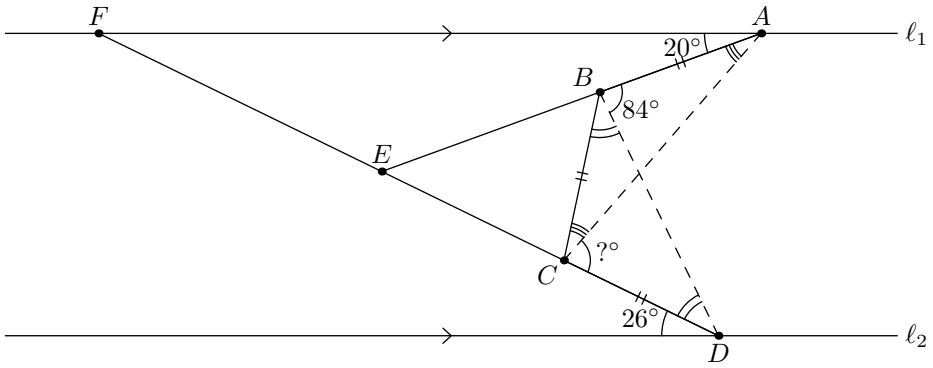
3. **[30]** Let points A and D lie on parallel lines ℓ_1 and ℓ_2, respectively, and let points B and C lie between ℓ_1 and ℓ_2 such that $AB = BC = CD$, and B, C are on the same side of AD. Suppose the angle between $\overline{AB}$ and ℓ_1 is $20°$, the angle between $\overline{CD}$ and ℓ_2 is $26°$, and $\angle ABD = 84°$. Compute the degree measure of $\angle ACD$.



Proposed by: Jason Mao

Answer: $\boxed{75°}$

Solution: Let E and F be the intersections of DC with AB and ℓ_1, respectively, as in the diagram below.



Note that

$$\angle BEC = \angle EAF + \angle EFA = 20° + 26° = 46°.$$

Hence,

$$\angle CBD = \angle CDB = \angle EDB = \angle ABD - \angle BED = 84° - 46° = 38°.$$

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This implies

$$\angle BCA = \frac{180^\circ - \angle ABC}{2} = \frac{180^\circ - \angle ABD - \angle CBD}{2} = \frac{180^\circ - 84^\circ - 38^\circ}{2} = 29^\circ.$$

Finally,

$$\begin{aligned}\angle ACD &= 180^\circ - \angle BCA - \angle BCE \\ &= 180^\circ - \angle BCA - \angle CBD - \angle CDB \\ &= 180^\circ - 29^\circ - 38^\circ - 38^\circ \\ &= \boxed{75^\circ}.\end{aligned}$$

4. **[35]** Let F_0, F_1, F_2, ... be a sequence of positive integers with $F_0 = F_1 = F_2 = \cdots = F_{2025} = 1$ and $F_i = F_{i-2025} + F_{i-2026}$ for all positive integers $i \geq 2026$. Compute the number of positive integers $n < 2026$ such that there exists a positive integer k satisfying $F_k = n$.

Proposed by: Sebastian Attlan

Answer: 64

Solution: For concreteness, the first several terms in the sequence are arranged in rows of 2025 terms below:

$$\begin{aligned}
a_{2025\cdot 0} &= 1 \quad 1 \quad \cdots \quad 1 &&= a_{2025\cdot 1-1} \\
a_{2025\cdot 1} &= 1 \overset{1}{\frown} 2 \quad 2 \quad \cdots \quad 2 &&= a_{2025\cdot 2-1} \\
a_{2025\cdot 2} &= 2 \overset{1}{\frown} 3 \overset{1}{\frown} 4 \quad 4 \quad \cdots \quad 4 &&= a_{2025\cdot 3-1} \\
a_{2025\cdot 3} &= 4 \overset{1}{\frown} 5 \overset{2}{\frown} 7 \overset{1}{\frown} 8 \quad 8 \quad \cdots \quad 8 &&= a_{2025\cdot 4-1} \\
a_{2025\cdot 4} &= 8 \overset{1}{\frown} 9 \overset{3}{\frown} 12 \overset{3}{\frown} 15 \overset{1}{\frown} 16 \quad 16 \quad \cdots \quad 16 &&= a_{2025\cdot 5-1} \\
a_{2025\cdot 5} &= 16 \overset{1}{\frown} 17 \overset{4}{\frown} 21 \overset{6}{\frown} 27 \overset{4}{\frown} 31 \overset{1}{\frown} 32 \quad 32 \quad \cdots \quad 32 &&= a_{2025\cdot 6-1} \\
a_{2025\cdot 6} &= 32 \overset{1}{\frown} 33 \overset{5}{\frown} 38 \overset{10}{\frown} 48 \overset{10}{\frown} 58 \overset{5}{\frown} 63 \overset{1}{\frown} 64 \quad 64 \quad \cdots \quad 64 &&= a_{2025\cdot 7-1} \\
a_{2025\cdot 7} &= 64 \overset{1}{\frown} 65 \overset{6}{\frown} 71 \overset{15}{\frown} 86 \overset{20}{\frown} 106 \overset{15}{\frown} 121 \overset{6}{\frown} 127 \overset{1}{\frown} 128 \quad 128 \quad \cdots \quad 128 &&= a_{2025\cdot 8-1}
\end{aligned}$$

Inspired by the table, we make three claims about the sequence (F_n). We will formally prove them at the end of the solution.

Claim 1. The sequence (F_i) is nondecreasing (that is, $F_{i+1} \geq F_i$ for all nonnegative integers i).

Claim 2. For all nonnegative integers $a \leq 2024$ and all integers b with $a \leq b \leq 2025$, $F_{2025a+b} = 2^a$. (In the table above, this refers to the long stretches of 2^a at the end of row a, in addition to the single 2^a at the beginning of row $a+1$.)

Claim 3. For all integers $1 \leq a \leq 2024$ and all integers b with $1 \leq b \leq a$, $F_{2025a+b} = 2^{a-1} + \sum_{i=0}^{b-1} \binom{a-1}{i}$. (In the table, this explains the entries in the triangle.)

Because the sequence is nondecreasing (Claim 1), once we encounter a term $F_n \geq 2026$, all future terms of the sequence will be at least 2026. We use Claim 2 and Claim 3 to pinpoint the smallest n

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with $F_n \geq 2026$. By Claim 2, $F_{2025(11)} = 2^{10} = 1024$, while $F_{2025(11)+11} = 2^{11} = 2048 > 2026$. By Claim 3, we have

$$F_{2025(11)+10} = 2^{10} + \sum_{i=0}^{9} \binom{10}{i} = 2^{10} + 2^{10} - \binom{10}{0} = 2047,$$

$$F_{2025(11)+9} = 2^{10} + \sum_{i=0}^{8} \binom{10}{i} = 2^{10} + 2^{10} - \binom{10}{0} - \binom{10}{1} = 2037,$$

$$F_{2025(11)+8} = 2^{10} + \sum_{i=0}^{7} \binom{10}{i} = 2^{10} + 2^{10} - \binom{10}{0} - \binom{10}{1} - \binom{10}{2} = 1992.$$

Hence, we only need to consider terms F_n with $0 \leq n \leq 2025(11) + 8$. We will count the number of distinct terms in this range.

For any nonnegative integer $1 \leq a \leq 2024$, by Claim 3, $F_{2025a+1} < \cdots < F_{2025a+a}$. By Claim 2, $F_{2025a+a} = F_{2025a+a+1} = \cdots = F_{2025(a+1)}$. Thus, between $F_{2025a+1}$ and $F_{2025(a+1)}$ inclusive, there are exactly a distinct values of F_n that occur. For $a = 0$, there is exactly 1 distinct value of F_n that occurs in the range F_1 to F_{2025} inclusive, namely 1. F_0 does not add any distinct values, as $F_0 = F_1$. Thus, the number of distinct values of F_n up to and including F_{2025a} is $1+1+2+3+\cdots+(a-1) = \frac{a(a-1)}{2}+1$. (Because the sequence is nondecreasing, there are no repeated values that arise from different values of a.)

It follows that there are $\frac{11(10)}{2} + 1 = 56$ distinct values of F_n up to and including $F_{2025(11)}$. Since $F_{2025(11)} < F_{2025(11)+1} < \cdots < F_{2025(11)+8}$, we need to add 8 more distinct values, for a total of $56 + 8 = \boxed{64}$.

We now provide formal proofs of the three claims.

Proof of Claim 1. We proceed by strong induction on i.

Base cases: the statement is true for all $0 \leq i \leq 2025$, because $F_i = 1$ for all $0 \leq i \leq 2026$.

Induction step: assume the statement is true for all i' with $i' \leq i$. Then

$$F_{i+1} = F_{i-2024} + F_{i-2025} \geq F_{i-2025} + F_{i-2026} = F_i.$$

where we have used the inductive hypothesis to claim $F_{i-2024} \geq F_{i-2025}$ and $F_{i-2025} \geq F_{i-2026}$. □

Proof of Claim 2. We proceed by induction on a.

Base case: the statement is true for $a = 0$, since $F_0 = \cdots = F_{2025} = 1$.

Induction step: suppose the statement is true for a and all integers b with $a \leq b \leq 2025$; we show it is true for $a + 1$ and any integer b satisfying $a + 1 \leq b \leq 2025$. Choose any such b. Then

$$F_{2025(a+1)+b} = F_{2025a+b} + F_{2025a+b-1} = 2^a + 2^a = 2^{a+1},$$

where we have used the inductive hypothesis twice, on (a, b) and $(a, b-1)$. □

Proof of Claim 3. We proceed by induction on a.

Base case: the statement is true for $a = 1$, since $F_{2025} = 2^{a-1} = 1$ and $F_{2026} = 2^{a-1} + \binom{0}{0} = 2$.

Inductive step: suppose the statement is true for a and all integers b with $0 \leq b \leq a$; we show it is true for $a + 1$ and any integer b satisfying $1 \leq b \leq a$. Choose any such b. If $b > 1$, then we can apply

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inductive hypothesis twice, on (a, b) and $(a, b-1)$:

$$\begin{aligned}
F_{2025(a+1)+b} &= F_{2025a+b} + F_{2025a+b-1} \\
&= 2^{a-1} + \sum_{i=0}^{b-1} \binom{a-1}{i} + 2^{a-1} + \sum_{i=0}^{b-2} \binom{a-1}{i} \\
&= 2^a + 1 + \sum_{i=1}^{b-1} \binom{a-1}{i} + \sum_{i=0}^{b-1} \binom{a-1}{i} \\
&= 2^a + 1 + \sum_{i=1}^{b-1} \left(\binom{a-1}{i} + \binom{a-1}{i-1} \right) \\
&= 2^a + 1 + \sum_{i=1}^{b-1} \binom{a}{i} && \text{(Pascal's identity)} \\
&= 2^a + \sum_{i=0}^{b-1} \binom{a}{i},
\end{aligned}$$

as needed.

If $b = 1$, the above argument doesn't work (since inductive hypothesis no longer applies to the pair of integers $(a, b-1)$, wherein $b = 0$). Instead, we use Claim 2 to obtain $F_{2025a+b-1} = F_{2025a} = F_{2025(a-1)+2025} = 2^{a-1}$. Using inductive hypothesis on (a, b), we can compute

$$\begin{aligned}
F_{2025(a+1)+b} &= F_{2025a+b} + F_{2025a+b-1} \\
&= F_{2025a+1} + F_{2025a} \\
&= 2^{a-1} + \sum_{i=0}^{0} \binom{a-1}{i} + 2^{a-1} \\
&= 2^a + 1 \\
&= 2^a + \sum_{i=0}^{b-1} \binom{a}{i},
\end{aligned}$$

as desired. □

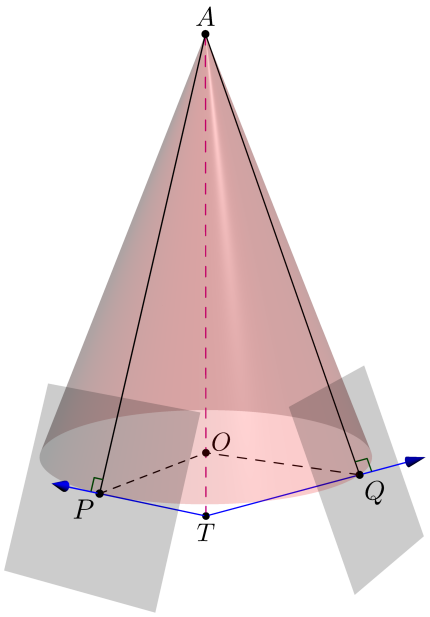
5. **[40]** A right circular cone with apex A has base radius 1 and height 7. Two perpendicular planes are tangent to the cone's lateral surface along segments $\overline{AP}$ and $\overline{AQ}$, where P and Q lie on the base of the cone. Compute PQ.

Proposed by: Jason Mao

Answer: $\boxed{\frac{10}{7}}$

Solution 1:

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Let T be the unique point (on the axis of the cone) such that for any point K on the circumference of the cone's base, we have $\angle AKT = 90°$. Then, the plane containing AP is orthogonal to PT, and the plane containing AQ is orthogonal to QT. Two planes are perpendicular if and only if their normal vectors are orthogonal, so the perpendicularity of the tangent planes is equivalent to the condition that $\angle PTQ = 90°$.

Note that for any point K on the circumference of the cone's base, KT is some fixed length λ. Let O be the center of the cone's base. Then, $\angle KAO = \angle TAK$ and $\angle AOK = \angle AKT = 90°$, so $\triangle AOK \sim \triangle AKT$. We can then compute

$$\lambda = KT = OK \cdot \frac{AK}{AO} = 1 \cdot \frac{\sqrt{1^2 + 7^2}}{7} = \frac{5\sqrt{2}}{7}.$$

In particular, $PT = QT = \frac{5\sqrt{2}}{7}$ and $\angle PTQ = 90°$, so

$$PQ = \sqrt{2} \cdot PT = \boxed{\frac{10}{7}}.$$

Solution 2: Define T as in solution 1, and let $\overrightarrow{XY}$ denote the vector from X to Y. As shown in solution 1, PT and QT are orthogonal, so

$$\overrightarrow{PT} \cdot \overrightarrow{QT} = 0.$$

Now, we will compute $\overrightarrow{PT}$ and $\overrightarrow{QT}$. Let the base lie in the xy-plane, and let $A = (0, 0, 7)$ lie on the z-axis. Without loss of generality, we may let $P = (1, 0, 0)$. Then, $Q = (\cos\theta, \sin\theta, 0)$ for some angle θ. Let $O = (0, 0, 0)$ be the origin, which is also the center of the base.

Let P' be the reflection of P over AT. Then, $\angle AP'T = \angle APT = 90°$, so P and P' both lie on the circle with diameter AT. As PP' and AT intersect at O, by Power of a Point,

$$OT = \frac{OP \cdot OP'}{OA} = \frac{1 \cdot 1}{7} = \frac{1}{7}.$$

Thus, $T = (0, 0, -\frac{1}{7})$.

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Then, we find

$$\overrightarrow{PT} = \left(-1, 0, -\frac{1}{7}\right),$$
$$\overrightarrow{QT} = \left(-\cos\theta, -\sin\theta, -\frac{1}{7}\right).$$

Therefore,

$$0 = \overrightarrow{PT} \cdot \overrightarrow{QT} = \left(-1, 0, -\frac{1}{7}\right) \cdot \left(-\cos\theta, -\sin\theta, -\frac{1}{7}\right) = \cos\theta + \frac{1}{49},$$

which gives $\cos\theta = -\frac{1}{49}$.

By Law of Cosines on $\triangle POQ$, we have

$$PQ^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \left(-\frac{1}{49}\right) = \frac{100}{49},$$

which gives $PQ = \boxed{\frac{10}{7}}$.

6. **[45]** Given that a and b are integers greater than 1 such that $\frac{a+b^2}{a^2}$ is an integer, compute the least possible value of b.

Proposed by: Can Erturan

Answer: $\boxed{35}$

Solution: We begin by proving the following claim.

Claim 1. a is a perfect square.

Proof. Write $\frac{a+b^2}{a^2} = x$ for some integers x. This rearranges to

$$b^2 = xa^2 - a = a(xa - 1).$$

Since $\gcd(a, xa-1)$ and $a(xa-1)$ is a perfect square. It follows that a is a perfect square. □

Thus, we let $a = c^2$ for some integer $c > 1$. Then, $c^4 \mid c^2 + b^2$, so $c^2 \mid c^2 + b^2$. This implies $c^2 \mid b^2$, so $b = nc$ for some integer n. This yields $c^4 \mid c^2 + n^2c^2$, which is equivalent to $c^2 \mid 1 + n^2$. We want to minimize nc.

We know n^2 is always 0 or 1 mod 4, so $1 + n^2$ is always 1 or 2 mod 4, and hence never divisible by 4. Thus, 4 cannot divide c^2, so c must be odd.

Likewise, n^2 is always 0 or 1 mod 3, so $1 + n^2$ is always 1 or 2 mod 3, and hence never divisible by 3, Thus, c cannot be divisible by 3 either.

As $c > 1$, the smallest possible values of c are 5 and 7. If $c = 5$, then $25 \mid 1 + n^2$, which implies $n \geq 7$; note $n = 7$ works as $1 + 7^2 = 50$. This yields $b = 5 \cdot 7 = 35$.

If $c \geq 7$, then since $n^2 + 1 \geq c^2$, we also have $n \geq 7$. Then, $n \geq 7 \cdot 7 = 49$.

Therefore, $b = nc \geq 35$. Observe that taking $n = 7$ and $c = 5$ gives $a = c^2 = 25$. This is indeed a solution, as

$$a + b^2 = 25 + 35^2 = 25(1 + 7^2) = 25 \cdot 50$$

is divisible by 25^2.

Thus, the answer is $\boxed{35}$.

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7. **[45]** Alice has 8 balls labeled 1, 2, 3, ..., 8 and 8 boxes labeled 1, 2, 3, ..., 8. Compute the number of ways for Alice to place exactly 1 ball in every box such that exactly 1 ball is labeled with a strictly higher number than its box.

Proposed by: Jacopo Rizzo

Answer: $\boxed{247}$

Solution: Note that each way for Alice to place the balls is equivalent to a permutation f from the set $\{1, 2, \ldots, 8\}$ to itself, by having Alice place ball i in box $f(i)$. Thus, we want to count the number of these permutations such that there is exactly one index i with $f(i) < i$. Call such an index *good*.

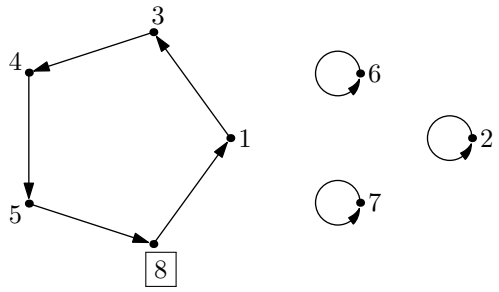
Suppose f has exactly one good index. Consider the cycle decomposition of f. Note f cannot be the identity, as the identity has no good indices, so f must have at least one cycle of length 2 or more. Take such a cycle $(a_1\, a_2 \cdots a_k)$. Without loss of generality, let a_1 be the smallest index in this cycle. Then, $f(a_k) = a_1 < a_k$, so a_k is a good index, and hence the only good index of f.

This implies that for $i < k$, we must have $f(a_i) \geq a_i$, i.e., $a_{i+1} \geq a_i$, so we have

$$a_1 \leq a_2 \leq \cdots \leq a_k.$$

Also, since every cycle of length at least 2 has at least one good index, f can only have one such cycle; the remaining indices must all be in cycles of length 1, i.e., fixed points.

Therefore, the cycle decomposition of f must be composed of one strictly increasing cycle of length at least 2, and all remaining cycles have length 1. Conversely, any such f has exactly one good index, as the increasing cycle of length at least 2 has exactly one good index, and the remaining fixed points are not good indices.



Example of a cycle decomposition of a permutation with only one good index, which is boxed.

Such a permutation f is uniquely determined by the elements in the single cycle of length at least 2, as they must then be in increasing order, and the remaining indices are fixed points. Thus, it suffices to count the number of subsets of $\{1, 2, \ldots, 8\}$ with size at least 2. The answer is

$$\sum_{k=2}^{8} \binom{8}{k} = 2^8 - 8 - 1 = \boxed{247.}$$

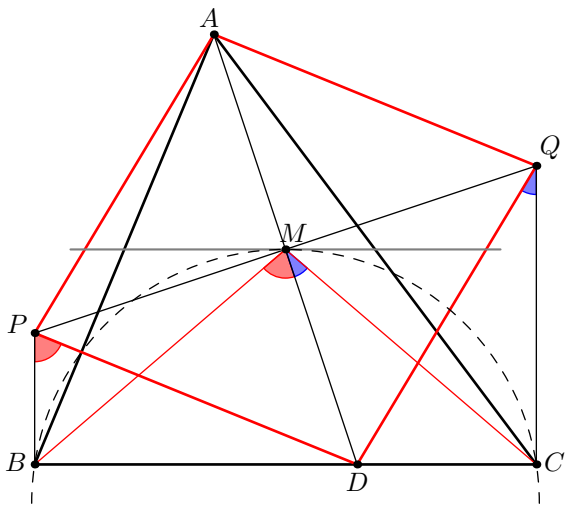
8. **[50]** Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Point D is chosen on segment BC. Points P and Q are such that $\angle PBC = 90°$, $\angle QCB = 90°$, and $APDQ$ is a kite with axis of symmetry PQ. Suppose that D is chosen such that $\angle PAQ$ is maximized. Compute AD.

Proposed by: Aprameya Tripathy

Answer: $\boxed{4\sqrt{10}.}$

Solution:

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Note by properties of kite that PQ is the perpendicular bisector of AD, and $\angle PAQ = \angle PDQ$. Let PQ intersect AD at M, midpoint of AD. Also note, as $\angle PMD = \angle PBD = 90°$, P, M, B, D and similarly, Q, M, C, D are cyclic. So,

$$\begin{aligned}\angle PAQ = \angle PDQ &= \angle PDM + \angle QDM \\ &= \angle PBM + \angle QCM \\ &= (90° - \angle MBC) + (90° - \angle MCB) \\ &= 180° - \angle MBC - \angle MCB \\ &= \angle BMC\end{aligned}$$

Thus, we want to maximize $\angle BMC$. However, M varies the fixed line ℓ joining midpoints of AB and AC. So, if the circumcircle of $\triangle BMC$ is not tangent to ℓ, there is a point M' on ℓ which is in the interior of said circumcircle. But then, $\angle BM'C$ is larger than $\angle BMC$. So, in order to maximize $\angle BMC$, the circumcircle of $\triangle BMC$ must be tangent to ℓ (at M). In particular, $BM = MC$.

Now, to conclude, note that by Heron's formula, the area of $\triangle ABC$ is $\sqrt{s(s-a)(s-b)(s-c)} = 84$, which gives that altitude along A has length $\frac{2 \cdot 84}{BC} = 12$.

Let the feet from A be A', then by Pythagoeran theorem, we have $BA' = \sqrt{BA^2 - AA'^2} = \sqrt{13^2 - 12^2} = 5$ and $CA' = BC - BA' = 14 - 5 = 9$. This means that the perpendicular bisector of BC, therefore also M is at a distance $\frac{14}{2} - 5 = 2$ from AA'. Then, as D is the reflection of A over M, we have $A'D = 2 \cdot 2 = 4$, so $AD^2 = AA'^2 + DA'^2 = 144 + 16 = 160$. Hence, the answer is $\sqrt{160} = \boxed{4\sqrt{10}}$.

9. **[50]** Let x, y, and z be real numbers satisfying

$$\begin{aligned}x + y - \sqrt{z^2 + 1} &= -5, \\ y + z - \sqrt{x^2 + 1} &= 0, \\ z + x - \sqrt{y^2 + 1} &= 5.\end{aligned}$$

Compute $x + y + z$.

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{\frac{\sqrt{38}+3\sqrt{2}}{2}}$.

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Solution: We make the substitution

$$x = \frac{1}{2}\left(a - \frac{1}{a}\right), \quad y = \frac{1}{2}\left(b - \frac{1}{b}\right), \quad z = \frac{1}{2}\left(c - \frac{1}{c}\right)$$

for some real $a, b, c > 0$. The point of this substitution is so that

$$\sqrt{x^2+1} = \frac{1}{2}\left(a + \frac{1}{a}\right), \quad \text{etc.}$$

Preforming this substitution and rearranging transforms the system of equations into

$$\begin{aligned}
\frac{1}{2}\left(a + b - c - \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\right) &= -5 \\
\frac{1}{2}\left(b + c - a - \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\right) &= 0 \\
\frac{1}{2}\left(c + a - b - \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\right) &= 5.
\end{aligned}$$

Let $s = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Then adding every pair of the above equations gives

$$a - s = 0, \quad b - s = -5, \quad c - s = 5.$$

Thus, $a = s$, $b = s - 5$, and $c = s + 5$. Therefore,

$$s = \frac{1}{s} + \frac{1}{s-5} + \frac{1}{s+5}.$$

Combining fractions, we get that

$$s = \frac{3s^2 - 25}{s^3 - 25s},$$

which gives $s^4 - 28s^2 + 25 = 0$, or

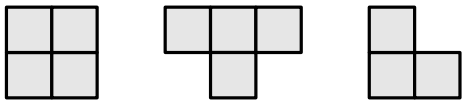
$$s = \sqrt{14 \pm 3\sqrt{19}} = \sqrt{\frac{28 \pm 2\sqrt{171}}{2}} = \frac{\sqrt{19} \pm 3}{\sqrt{2}}.$$

We have $a + b + c = 3s$, so

$$x + y + z = \frac{1}{2}(a + b + c) - \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = s.$$

Since $x + y + z > 3$ follows from adding the three equations in the problem statement, we get that the answer is $\boxed{\frac{\sqrt{38}+3\sqrt{2}}{2}}$.

10. **[55]** Let N denote the number of ways to tile a 2×199 grid with O-tetrominoes, T-tetrominoes, and L-trominoes, as shown below. Compute the remainder when N is divided by 128. (The tiles can be rotated, and tilings that differ by rotation or reflection are considered distinct.)



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Proposed by: Andrew Brahms

Answer: 44

Solution 1:

The main idea is to consider the possible placements of L-tiles on the grid first and then let the O- and T-tiles uniquely fill in the rest of the grid. Here is an example placement on a 2×21 grid.



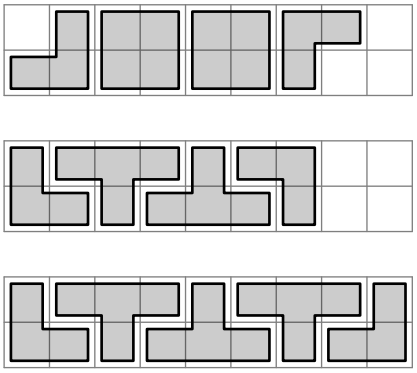
For any L-tile, consider the empty cell needed to complete it into an O. Call the tile *left-empty* or *right-empty* depending on whether the empty cell lies on the left or right (respectively) of the tile.

Call two L-tiles *consecutive* if there are no other L-tiles between them.

Claim 1. Two consecutive L-tiles cannot be both left or both right.

Proof. We consider cases based on the left L-tile's orientation.

- If the left L-tile is left-empty, then the tiles between the two L-tiles must all be O-tiles. Thus, the right L-tile must be right-empty to fit next to these O-tiles. (This case corresponds to the first case in the diagram below.)
- If the left L-tile is right-empty, then the tiles between the two L-tiles must all be T-tiles. Thus, the right L-tile must be left-empty to fit next to these T-tiles. (This case corresponds to the other two cases in the diagram below.)



□

Now note that the leftmost L-tile must be right-empty, as the tiles on its left can only be O-tiles. Similarly, the last L-tile must be left-empty, as the tiles on its right can only be O-tiles. Thus, we can pair up L's into consecutive pairs, each consisting of a right-empty L-tile on the left and a left-empty L-tile on the right. Only T-tiles appear in the two L-tiles in a single pair, and only O-tiles appear between the pairs. (The red rectangles outlined in the first diagram depict the pairs.)

Claim 2. The two L-tiles in a single pair bound a $2 \times m$ rectangle for some odd integer m (including the L-tiles themselves), and there is a unique way to fill the space between them with T-tiles.

Proof. Recall from the proof of the first claim that this space can only be filled with T-tiles. The T-tile directly on the right of the left L-tile in the pair is fixed, as there is only one way to fill the "hole" left by this L-tile. The next T-tile is then also fixed to fill the hole left by the first T-tile. This pattern continues until we reach the other L-tile in the pair, so the space between the two L-tiles can be filled in only one way.

If k T-tiles are used, then the rectangle bounded by the two L-tiles has area $4k + 6$, so $m = 2k + 3$ is odd. □

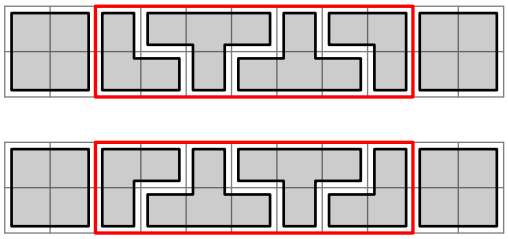
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Claim 3. Two consecutive L-tiles which are not in the same pair bound a $2 \times m$ rectangle between them for some even integer m (not including the L-tiles themselves), and there is a unique way to fill this space with O-tiles.

Proof. Like in the previous claim, only O-tiles can appear in this region, so it can clearly only be filled in one way. If k O-tiles are used, then the rectangle between the two L-tiles has area $4k$, so $m = 2k$ is even. □

Label the columns 1, 2, ..., 199 from left to right. Say the *position* of an L-tile is the column its leftmost cell occupies. By the previous two claims, two L-tiles in the same pair have positions of opposite parities, while two consecutive L-tiles not in a pair have positions of the same parity.

Now suppose there are ℓ pairs of L-tiles. After fixing the positions of the L-tiles, there are 2 ways to orient each rectangular block bounded by a pair of L-tiles, so there are 2^ℓ possible tilings with these fixed positions.



Now, we count the number of ways to pick positions for the L-tiles, given that there are ℓ pairs of them. Let their positions be a_1, b_1, a_2, b_2, ..., a_ℓ, b_ℓ from left to right. Then, the previous claims imply that for all i,

$$b_i - a_i \equiv 1 \pmod 2 \quad \text{and} \quad a_{i+1} - b_i \equiv 0 \pmod 2.$$

Thus, a_i and a_{i+1} have opposite parities. Furthermore, as only O-tiles can appear on the left of the leftmost L-tile, a_1 is odd. Likewise, b_ℓ is even, so a_ℓ is also odd. Thus, a_1 and a_ℓ have the same parity, implying ℓ is odd.

For $1 \le i \le \ell$ and $0 \le j \le \ell$, define the quantities

$$c_i = \frac{b_i - a_i - 1}{2} \quad \text{and} \quad d_j = \frac{a_{j+1} - b_j - 2}{2},$$

where we take $b_0 = -1$ and $a_{\ell+1} = 200$. (These are the number of tiles which appear between each two consecutive L-tiles, as well as on the left of the leftmost L-tile and on the right of the rightmost L-tile.) Then each c_i and d_j is a nonnegative integer, and

$$d_0 + c_1 + d_1 + \cdots + c_\ell + d_\ell = \frac{(a_{\ell+1} - b_0) - 2(\ell+1) - \ell}{2} = \frac{199 - 3\ell}{2}.$$

By stars and bars the number of ways to pick the c_i's and d_j's (which is $2\ell + 1$ values total) is

$$\binom{\frac{199+\ell}{2}}{2\ell}.$$

Each such choice can be tiled, as the parity constraints ensure that the regions between L-tiles can be filled by the appropriate tetrominoes. Thus, each such choice yields 2^ℓ possible tilings.

Summing over odd ℓ, the final count is then

$$\sum_{\text{odd } \ell \ge 1} 2^\ell \binom{\frac{199+\ell}{2}}{2\ell} = 2^1\binom{100}{2} + 2^3\binom{101}{6} + 2^5\binom{102}{10} + \cdots.$$

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Modulo 128, all terms after the first three vanish. In fact, we can also check the second and third terms vanish by checking which powers of 2 divide each binomial coefficient. We check

$$\nu_2\left(\binom{101}{6}\right) = \nu_2(101!) - \nu_2(6!) - \nu_2(95!) = 97 - 4 - 89 = 4$$

and

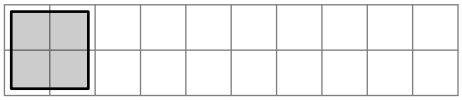
$$\nu_2\left(\binom{102}{10}\right) = \nu_2(101!) - \nu_2(10!) - \nu_2(91!) = 97 - 8 - 86 = 3,$$

so $2^7 = 128$ divides both $2^3\binom{101}{6}$ and $2^5\binom{102}{10}$.

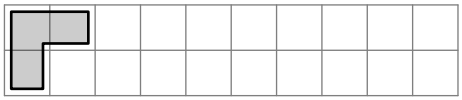
Thus, the answer is the remainder when $2^1\binom{100}{2} = 100 \cdot 99 = 9900$ is divided by 128, which is $\boxed{44}$.

Solution 2: We will compute the generating function for the number of ways to tile a $2 \times n$ grid. Let a_n be the number of ways to tile a $2 \times n$ grid and b_n be the number of ways to tile a $2 \times n$ grid with one of the cells of the leftmost column removed. We wish to find recurrence relations for a_n and b_n. We compute a_n first. In a tiling of a $2 \times n$ grid, the leftmost tile must be an O-tile or L-tile. The former case leaves a $2 \times (n-2)$ rectangle, giving us a_{n-2} tilings. In the latter case, the L-tile can be placed in two ways, each leaving a $2 \times (n-1)$ rectangle with one cell removed, yielding $2b_{n-1}$ tilings total. Thus,

$$a_n = a_{n-2} + 2b_{n-1}.$$

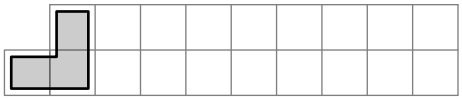


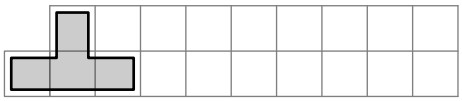




We now compute b_n. When tiling a $2 \times n$ grid with one of the leftmost cells removed, the leftmost tile must be a T-tile or L-tile. The former case leaves a $2 \times (n-2)$ rectangle, giving us a_{n-2} tilings. The latter case leaves a $2 \times (n-2)$ rectangle with one cell removed, giving us b_{n-2} tilings. Thus,

$$b_n = a_{n-2} + b_{n-2}.$$





We get a recurrence for a_n as follows. By the recurrence for b_{n-1},

$$a_{n-3} = b_{n-1} - b_{n-3},$$

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so we take the recurrence for a_{n-2} and subtract it from the recurrence for a_n to make use of this expression:

$$\begin{aligned}
&(a_n - a_{n-2} - 2b_{n-1}) - (a_{n-2} - a_{n-4} - 2b_{n-3}) = 0 \\
\implies &a_n - 2a_{n-2} + a_{n-4} - 2(b_{n-1} - b_{n-3}) = 0 \\
\implies &a_n - 2a_{n-2} + a_{n-4} - 2a_{n-3} = 0 \\
\implies &a_n - 2a_{n-2} - 2a_{n-3} + a_{n-4} = 0
\end{aligned}$$

This is a degree-4 linear recurrence. To get the first 3 terms, we use the initial terms $a_0 = 1$, $a_1 = 0$, $b_0 = 0$, and $b_1 = 0$ and the two sequence recurrence:

n	a_n	b_n
0	1	0
1	0	0
2	1	1
3	2	

Now, the characteristic polynomial of the linear recurrence for a_n is $x^4 - 2x^2 - 2x + 1$, so its generating function is of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots = \frac{P(x)}{1 - 2x^2 - 2x^3 + x^4}$$

for a polynomial $P(x)$ of degree at most 3. We now compute $P(x)$ using the initial terms:

$$\begin{aligned}
&\frac{P(x)}{1 - 2x^2 - 2x^3 + x^4} \equiv 1 + x + 2x^3 \pmod{x^4} \\
\implies &P(x) \equiv (1 + x + 2x^3)(1 - 2x^2 - 2x^3 + x^4) \pmod{x^4} \\
\implies &P(x) \equiv 1 - x^2 \pmod{x^4} \\
\implies &P(x) = 1 - x^2
\end{aligned}$$

It remains to compute the x^{199} term in $\frac{1-x^2}{1-x^2-2x^3+x^4}$ modulo 128. The key manipulation is to rewrite the generating function as

$$\begin{aligned}
\frac{1 - x^2}{1 - x^2 - 2x^3 + x^4} &= \frac{1 - x^2}{(1 - x^2)^2 - 2x^3} \\
&= \frac{\frac{1}{1-x^2}}{1 - \frac{2x^3}{(1-x^2)^2}} \\
&= \frac{1}{1 - x^2} + \frac{2x^3}{(1 - x^2)^3} + \frac{4x^6}{(1 - x^2)^5} + \cdots
\end{aligned}$$

so that all but the first 7 terms vanish modulo 128. Furthermore, for any m and n, $\frac{1}{(1-x^2)^n}$ only has terms of even exponent, so $\frac{x^m}{(1-x^2)^n}$ only has terms of exponent with the same parity as m.

Thus, only the odd exponent terms contribute to the coefficient of x^{199}. The coefficient of x^{2n} in $\frac{1}{(1-x^2)^k}$ is $\binom{n+k-1}{k-1}$, so the coefficient of x^{199} is

$$2^1 \binom{\frac{199-3}{2} + 2}{2} + 2^3 \binom{\frac{199-9}{2} + 6}{6} + 2^5 \binom{\frac{199-15}{2} + 10}{10} + \cdots = 2^1 \binom{100}{2} + 2^3 \binom{101}{6} + 2^5 \binom{102}{10} + \cdots.$$

Following the final computation in Solution 1, we once again find the answer is $\boxed{44}$.

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